

Integrals evaluated in terms of Catalan's constant

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Catalan's constant, named after E. C. Catalan (1814–1894) and usually denoted by G , is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots$$

It is, of course, a close relative of

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}.$$

The numerical value is $G \approx 0.9159656$. It is not known whether G is irrational: this remains a stubbornly unsolved problem. The best hope for a solution might appear to be the method of Beukers [1] to prove the irrationality of $\zeta(2)$ directly from the series, but it is not clear how to adapt this method to G .

A remarkable assortment of seemingly very different definite integrals equate to G , or are evaluated in terms of G . A compilation of no fewer than eighty integral and series representations for G , with proofs, is given by Bradley [2]. Another compilation, based on *Mathematica*, is [3]. Here we will present a selection of some of the simpler integrals and double integrals that are evaluated in terms of G , including a few that are actually not to be found in [2] or [3].

The most basic one is

$$\int_0^1 \frac{\tan^{-1} x}{x} dx = G, \tag{1}$$

obtained at once by termwise integration of the series

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}.$$

Termwise integration (for those who care) is easily justified. Write

$$s_{2n}(x) = \sum_{r=0}^n (-1)^r \frac{x^{2r}}{2r+1}.$$

Since the series is alternating, with terms decreasing in magnitude, we have $\tan^{-1} x/x = s_{2n}(x) + r_{2n}(x)$, where $|r_{2n}(x)| \leq x^{2n+2}/(2n+3)$, so that $\int_0^1 r_{2n}(x) dx \rightarrow 0$ as $n \rightarrow \infty$.

We now embark on a round trip of integrals derived directly from (1). Most of them can be seen in [4], but we repeat them here for completeness. First, the substitution $x = \tan \theta$

gives

$$G = \int_0^{\pi/4} \frac{\theta}{\tan \theta} \sec^2 \theta d\theta = \int_0^{\pi/4} \frac{\theta}{\sin \theta \cos \theta} d\theta. \quad (2)$$

Since $\tan \theta + \cot \theta = 1/(\sin \theta \cos \theta)$, (2) can be rewritten

$$\int_0^{\pi/4} \theta(\tan \theta + \cot \theta) d\theta = G. \quad (3)$$

The substitution $\theta = 2\phi$ gives one of the most important equivalent forms:

$$\int_0^{\pi/2} \frac{\theta}{\sin \theta} d\theta = \int_0^{\pi/4} \frac{4\phi}{\sin 2\phi} d\phi = \int_0^{\pi/4} \frac{2\phi}{\sin \phi \cos \phi} d\phi = 2G. \quad (4)$$

We mention in passing that (4) can be rewritten in terms of the gamma function. Recall that by Euler's reflection formula, $\Gamma(1+x)\Gamma(1-x) = \pi x/(\sin \pi x)$. Substituting $\pi x = \theta$ in (4), we deduce:

$$\int_0^{1/2} \Gamma(1+x)\Gamma(1-x) dx = \frac{2G}{\pi}.$$

The substitution $x = e^{-t}$ in (1) gives at once

$$G = \int_0^{\infty} \tan^{-1}(e^{-t}) dt. \quad (5)$$

Next, we integrate by parts in (1), obtaining

$$G = \left[\tan^{-1} x \log x \right]_0^1 - \int_0^1 \frac{\log x}{1+x^2} dx.$$

The first term is zero, since $\tan^{-1} x \log x \sim x \log x \rightarrow 0$ as $x \rightarrow 0^+$, hence

$$\int_0^1 \frac{\log x}{1+x^2} dx = -G. \quad (6)$$

Now substituting $x = 1/y$, we deduce

$$\int_1^{\infty} \frac{\log x}{1+x^2} dx = G. \quad (7)$$

Substituting $x = e^y$ in (7), we obtain

$$G = \int_0^{\infty} \frac{y}{1+e^{2y}} e^y dy = \int_0^{\infty} \frac{y}{2 \cosh y} dy,$$

hence

$$\int_0^{\infty} \frac{x}{\cosh x} dx = 2G. \quad (8)$$

Now substituting $x = \tan \theta$ in (6), we obtain

$$-G = \int_0^{\pi/4} \frac{\log \tan \theta}{\sec^2 \theta} \sec^2 \theta d\theta = \int_0^{\pi/4} \log \tan \theta d\theta. \quad (9)$$

Since $(1 + \cos \theta)/(1 - \cos \theta) = \cot^2 \frac{\theta}{2}$, we can deduce

$$\int_0^{\pi/2} \log \frac{1 + \cos \theta}{1 - \cos \theta} d\theta = -2 \int_0^{\pi/2} \log \tan \frac{\theta}{2} d\theta = -4 \int_0^{\pi/4} \log \tan \phi d\phi = 4G. \quad (10)$$

This, in turn, leads to an interesting series representation. Given the series

$$\frac{1}{2} \log \frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

and the well-known identity

$$\int_0^{\pi/2} \cos^{2n+1} \theta d\theta = \int_0^{\pi/2} \sin^{2n+1} \theta d\theta = \frac{2.4 \dots (2n)}{1.3 \dots (2n+1)}, \quad (11)$$

we deduce

$$2G = \sum_{n=0}^{\infty} \frac{2.4 \dots (2n)}{1.3 \dots (2n-1)(2n+1)^2} = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!(2n+1)^2} = \sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{2n}{n}(2n+1)^2}. \quad (12)$$

Using (11) again (but none of the earlier integrals), we now establish

$$\int_0^{\pi/2} \sinh^{-1}(\sin \theta) d\theta = G. \quad (13)$$

Write

$$a_n = \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)}.$$

By the binomial series,

$$\frac{1}{(1+x^2)^{1/2}} = \sum_{n=0}^{\infty} (-1)^n a_n x^{2n}.$$

Hence

$$\sinh^{-1} x = \int_0^x \frac{1}{(1+t^2)^{1/2}} dt = \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{2n+1} x^{2n+1}.$$

As above,

$$\int_0^{\pi/2} \sin^{2n+1} \theta d\theta = \frac{1}{a_n(2n+1)},$$

so by a neat cancellation of a_n ,

$$\int_0^{\pi/2} \sinh^{-1}(\sin \theta) d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G.$$

We remark that the same method, applied to $\sin^{-1}(\sin \theta)$ ($= \theta$) delivers the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \int_0^{\pi/2} \theta d\theta = \frac{\pi^2}{8}.$$

Integrals of logsine type and applications

A further application of (9) is to integrals of the logsine type, of which there are a rich variety. We start by stating the most basic such integral, which does not involve G :

$$\int_0^{\pi/2} \log \sin \theta \, d\theta = \int_0^{\pi/2} \log \cos \theta \, d\theta = -\frac{\pi}{2} \log 2. \quad (14)$$

Our proof follows [5, p. 246]). The substitution $\theta = \frac{\pi}{2} - \phi$ shows that the two integrals are equal: denote them both by I . Also, the substitution $\theta = \pi - \phi$ gives $\int_{\pi/2}^{\pi} \log \sin \theta \, d\theta = I$. Hence $2I = \int_0^{\pi} \log \sin \theta \, d\theta$. Now substituting $\theta = 2\phi$ and using $\sin 2\phi = 2 \sin \phi \cos \phi$, we have

$$\begin{aligned} 2I &= 2 \int_0^{\pi/2} \log \sin 2\phi \, d\phi \\ &= 2 \int_0^{\pi/2} (\log \sin \phi + \log \cos \phi + \log 2) \, d\phi \\ &= 4I + \pi \log 2, \end{aligned}$$

hence (14).

Note that the equality of the two integrals in (14) (without knowing their value) implies that $\int_0^{\pi/2} \log \tan \theta \, d\theta = 0$. Also, (14) can be restated neatly as follows:

$$\int_0^{\pi/2} \log(2 \sin \theta) \, d\theta = \int_0^{\pi/2} \log(2 \cos \theta) \, d\theta = 0.$$

The integral (14) has numerous equivalent forms. For example, first substituting $x = \sin \theta$ and then integrating by parts, we find

$$\int_0^1 \frac{\sin^{-1} x}{x} \, dx = \int_0^{\pi/2} \theta \cot \theta \, d\theta = - \int_0^{\pi/2} \log \sin \theta \, d\theta = \frac{\pi}{2} \log 2. \quad (15)$$

Further equivalents, and a survey of Euler's work in this area, are given in [6].

Catalan's constant enters the scene when we integrate from 0 to $\frac{\pi}{4}$ instead of $\frac{\pi}{2}$. Let

$$I_S = \int_0^{\pi/4} \log \sin \theta \, d\theta, \quad I_C = \int_0^{\pi/4} \log \cos \theta \, d\theta.$$

Substituting $\theta = \frac{\pi}{2} - \phi$, we have $I_C = \int_{\pi/4}^{\pi/2} \log \sin \theta \, d\theta$. So by (14), $I_S + I_C = \int_0^{\pi/2} \log \sin \theta \, d\theta = -\frac{\pi}{2} \log 2$. Meanwhile by (9), $I_S - I_C = \int_0^{\pi/4} \log \tan \theta \, d\theta = -G$. So we conclude

$$I_S = -\frac{1}{2}G - \frac{\pi}{4} \log 2, \quad I_C = \frac{1}{2}G - \frac{\pi}{4} \log 2. \quad (16)$$

Again there is a neat restatement:

$$\int_0^{\pi/4} \log(2 \sin \theta) \, d\theta = -\frac{1}{2}G, \quad \int_0^{\pi/4} \log(2 \cos \theta) \, d\theta = \frac{1}{2}G.$$

However, (16) will be more useful in the ensuing applications.

Alternatively, (14) and (16) can be derived from the series $\log(2 \sin \theta) = -\sum_{n=1}^{\infty} \frac{1}{n} \cos 2n\theta$, $\log(2 \cos \theta) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos 2n\theta$ (e.g. see [7]); however, justification of termwise convergence is more delicate in this case.

The substitution $x = \tan \theta$ delivers the following reformulation of (16) in terms of the log function:

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2} dx = \int_0^{\pi/4} \log \sec^2 \theta d\theta = -2 \int_0^{\pi/4} \log \cos \theta d\theta = \frac{\pi}{2} \log 2 - G. \quad (17)$$

One can easily check that by substituting $x = 1/y$ and combining with (7), one obtains

$$\int_1^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = G + \frac{\pi}{2} \log 2. \quad (18)$$

Now integrate by parts in (16): we find

$$\begin{aligned} \int_0^{\pi/4} \log \cos \theta d\theta &= \left[\theta \log \cos \theta \right]_0^{\pi/4} + \int_0^{\pi/4} \theta \frac{\sin \theta}{\cos \theta} d\theta \\ &= -\frac{\pi}{8} \log 2 + \int_0^{\pi/4} \theta \tan \theta d\theta, \end{aligned}$$

so that

$$\int_0^{\pi/4} \theta \tan \theta d\theta = \frac{1}{2}G - \frac{\pi}{8} \log 2, \quad (19)$$

an identity not given in [2] or [3]. Similarly, or by (19) combined with (3), we have $\int_0^{\pi/4} \theta \cot \theta d\theta = \frac{1}{2}G + \frac{\pi}{8} \log 2$.

Next, write

$$I = \int_0^{\pi/2} \log(1 + \sin \theta) d\theta = \int_0^{\pi/2} \log(1 + \cos \theta) d\theta.$$

By (16) and the identity $1 + \cos \theta = 2 \cos^2 \frac{1}{2}\theta$, we obtain

$$\begin{aligned} I &= \int_0^{\pi/2} (\log 2 + 2 \log \cos \frac{1}{2}\theta) d\theta \\ &= \frac{\pi}{2} \log 2 + 4 \int_0^{\pi/4} \log \cos \phi d\phi \\ &= \frac{\pi}{2} \log 2 + 2G - \pi \log 2 \\ &= 2G - \frac{\pi}{2} \log 2, \end{aligned} \quad (20)$$

another integral not given in [2] or [3]. Combining (20) and (14), we have the pleasingly simple result

$$\int_0^{\pi/2} \log(1 + \operatorname{cosec} \theta) d\theta = \int_0^{\pi/2} \left(\log(1 + \sin \theta) - \log \sin \theta \right) d\theta = 2G, \quad (21)$$

and of course the same applies with $\operatorname{cosec} \theta$ replaced by $\sec \theta$.

Writing $(1 + \sin \theta)(1 - \sin \theta) = \cos^2 \theta$, we deduce further

$$\begin{aligned} \int_0^{\pi/2} \log(1 - \sin \theta) d\theta &= 2 \int_0^{\pi/2} \log \cos \theta d\theta - \int_0^{\pi/2} \log(1 + \sin \theta) d\theta \\ &= -2G - \frac{\pi}{2} \log 2. \end{aligned} \quad (22)$$

Integrating by parts in (20), we have

$$\int_0^{\pi/2} \log(1 + \sin \theta) d\theta = \left[\theta \log(1 + \sin \theta) \right]_0^{\pi/2} - J = \frac{\pi}{2} \log 2 - J,$$

where

$$J = \int_0^{\pi/2} \frac{\theta \cos \theta}{1 + \sin \theta} d\theta,$$

hence

$$J = \pi \log 2 - 2G. \quad (23)$$

Furthermore, the substitution $x = \sin \theta$ gives

$$\int_0^1 \frac{\sin^{-1} x}{1 + x} dx = J. \quad (24)$$

A further deduction from (16) is derived using the identity $\cos \theta + \sin \theta = \sqrt{2} \cos(\theta - \frac{\pi}{4})$:

$$\begin{aligned} \int_0^{\pi/2} \log(\cos \theta + \sin \theta) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2} \log 2 + \log \cos(\theta - \frac{\pi}{4}) \right) d\theta \\ &= \frac{\pi}{4} \log 2 + \int_{-\pi/4}^{\pi/4} \log \cos \phi d\phi \\ &= \frac{\pi}{4} \log 2 + G - \frac{\pi}{2} \log 2 \\ &= G - \frac{\pi}{4} \log 2. \end{aligned} \quad (25)$$

Alternatively, (25) can be deduced from (20) and the identity $(\cos \theta + \sin \theta)^2 = 1 + \sin 2\theta$.

By (25) and (14),

$$\begin{aligned} \int_0^{\pi/2} \log(1 + \tan \theta) d\theta &= \int_0^{\pi/2} \left(\log(\cos \theta + \sin \theta) - \log \cos \theta \right) d\theta \\ &= G - \frac{\pi}{4} \log 2 + \frac{\pi}{2} \log 2 \\ &= G + \frac{\pi}{4} \log 2, \end{aligned} \quad (26)$$

and hence also, with the substitution $x = \tan \theta$,

$$\int_0^\infty \frac{\log(1+x)}{1+x^2} dx = G + \frac{\pi}{4} \log 2, \quad (27)$$

which is proved by a rather longer method in [2].

Of course, $\tan \theta$ can be replaced by $\cot \theta$ in (26). With (20) and (21), this means that we have obtained the values of $\int_0^{\pi/2} \log[1+f(\theta)] d\theta$, where $f(\theta)$ can be any one of the six trigonometric functions.

Yet further integrals can be derived by integrating by parts in (25) and (26) (rather better with $\cot \theta$): we leave it to the reader to explore this..

Double integrals

An obvious double-integral representation of G , not involving any trigonometric or logarithmic functions, follows at once from the geometric series. For $x, y \in [0, 1)$, we have $1/(1+x^2y^2) = \sum_{n=0}^\infty (-1)^n x^{2n} y^{2n}$. Since

$$\int_0^1 \int_0^1 x^{2n} y^{2n} dx dy = \frac{1}{(2n+1)^2},$$

termwise integration gives

$$\int_0^1 \int_0^1 \frac{1}{1+x^2y^2} dx dy = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2} = G. \quad (28)$$

Termwise integration is justified as in (1). (Alternatively, one stage of integration immediately equates the double integral to (1)).

Substituting $x = e^{-u}$ and $y = e^{-v}$, we deduce

$$G = \int_0^\infty \int_0^\infty \frac{e^{-u} e^{-v}}{1+e^{-2u} e^{-2v}} du dv = \int_0^\infty \int_0^\infty \frac{1}{2 \cosh(u+v)} du dv. \quad (29)$$

(This is really a pair of successive single-variable substitutions, not a full-blooded two-variable one.)

Next, we establish a much less transparent double-integral representation:

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{1+\cos \theta \cos \phi} d\theta d\phi = 2G. \quad (30)$$

This, again, is not in [2] or [3]. It was given in [4], possibly its first appearance. Here we present a proof based on (2). We actually show

$$\int_0^{\pi/4} \int_0^{\pi/4} \frac{1}{1+\cos 2\theta \cos 2\phi} d\theta d\phi = \frac{1}{2}G, \quad (31)$$

from which (30) follows by substituting $\theta = 2\theta'$ and $\phi = 2\phi'$. Write

$$J(\theta) = \int_0^{\pi/4} \frac{1}{1 + \cos 2\theta \cos 2\phi} d\phi.$$

Now

$$\begin{aligned} 1 + \cos 2\theta \cos 2\phi &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \phi + \sin^2 \phi) + (\cos^2 \theta - \sin^2 \theta)(\cos^2 \phi - \sin^2 \phi) \\ &= 2(\cos^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi). \end{aligned} \quad (32)$$

The substitution $t = \tan \phi$ gives

$$\int_0^{\pi/4} \frac{1}{a^2 \cos^2 \phi + b^2 \sin^2 \phi} d\phi = \frac{1}{ab} \tan^{-1} \frac{b}{a}.$$

Applied with $a = \cos \theta$ and $b = \sin \theta$, this gives

$$J(\theta) = \frac{\theta}{2 \sin \theta \cos \theta}.$$

By (2), it follows that $\int_0^{\pi/4} J(\theta) d\theta = \frac{1}{2}G$, which is (31).

Alternatively, substitute $x = \tan \theta$ and $y = \tan \phi$ in (28). Again, (31) is obtained after applying (32). (This is the method of [4].)

We now establish two further double integrals:

$$\int_0^1 \int_0^1 \frac{1}{2 - x^2 - y^2} dx dy = G, \quad (33)$$

$$\int_0^1 \int_0^1 \frac{1}{(x+y)(1-x)^{1/2}(1-y)^{1/2}} dx dy = 4G. \quad (34)$$

Result (34) is identity (44) in [2], achieved with some effort; we present a somewhat simpler proof. The substitution $x = 1 - u^2$, $y = 1 - v^2$ reduces (34) to (33). Denote the integral in (33) by I . Since the integrand satisfies $f(y, x) = f(x, y)$, the contributions with $x \leq y$ and $y \leq x$ are the same, hence

$$I = 2 \int_0^1 \int_0^x \frac{1}{2 - x^2 - y^2} dy dx.$$

Transform to polar coordinates:

$$I = \int_0^{\pi/4} \int_0^{\sec \theta} \frac{2r}{2 - r^2} dr d\theta.$$

Now

$$\int_0^{\sec \theta} \frac{2r}{2 - r^2} dr = \left[-\log(2 - r^2) \right]_0^{\sec \theta} = \log \frac{2}{2 - \sec^2 \theta}$$

and

$$\frac{2}{2 - \sec^2 \theta} = \frac{2}{1 - \tan^2 \theta} = \frac{\tan 2\theta}{\tan \theta},$$

so

$$I = \int_0^{\pi/4} (\log \tan 2\theta - \log \tan \theta) d\theta.$$

As seen in (14), $\int_0^{\pi/4} \log \tan 2\theta d\theta = \frac{1}{2} \int_0^{\pi/2} \log \tan \phi d\phi = 0$. By (9), we deduce that $I = G$.

The discussion site [8] gives the following variant of (33):

$$\int_0^1 \int_0^1 \frac{1}{4 - x^2 - y^2} dx dy = \frac{1}{3}G.$$

The method is similar, but the writer resorts to *Mathematica* for the final stage, the integral $\int_0^{\pi/4} \log[4/(4 - \sec^2 \theta)] d\theta$; with a bit of effort, this can be deduced from our results above using the identity $4/(4 - \sec^2 \theta) = 2 \sin 2\theta \cos \theta / (\sin 3\theta)$.

Applications to elliptic integrals

The “complete elliptic integral of the first kind” is defined, for $0 \leq t < 1$, by

$$K(t) = \int_0^{\pi/2} \frac{1}{(1 - t^2 \sin^2 \theta)^{1/2}} d\theta.$$

The value of $K(t)$ is given explicitly by a theorem of Gauss:

$$K(t) = \frac{\pi}{2M(1, t^*)},$$

where $M(a, b)$ is the arithmetic-geometric mean of a and b , and $t^* = (1 - t^2)^{1/2}$. Two quite different proofs of this theorem can be seen in [9] and [10]. However, without any reference to Gauss’s theorem, we can apply (4) to evaluate $\int_0^1 K(t) dt$. Indeed, reversing the implied double integral, we have $\int_0^1 K(t) dt = \int_0^{\pi/2} F(\theta) d\theta$, where

$$F(\theta) = \int_0^1 \frac{1}{(1 - t^2 \sin^2 \theta)^{1/2}} dt.$$

Substituting $t \sin \theta = \sin \phi$, we have

$$F(\theta) = \int_0^\theta \frac{1}{\cos \phi} \frac{\cos \phi}{\sin \theta} d\phi = \frac{\theta}{\sin \theta}.$$

Hence, by (4),

$$\int_0^1 K(t) dt = 2G. \tag{35}$$

Of course, this really amounts to another double-integral representation of G .

The “complete elliptic integral of the second kind” is

$$E(t) = \int_0^{\pi/2} (1 - t^2 \sin^2 \theta)^{1/2} d\theta.$$

In the same way, we have $\int_0^1 E(t) dt = \int_0^{\pi/2} G(\theta) d\theta$, where

$$\begin{aligned} G(\theta) &= \int_0^1 (1 - t^2 \sin^2 \theta)^{1/2} dt \\ &= \int_0^\theta \cos \phi \frac{\cos \phi}{\sin \theta} d\phi \\ &= \frac{1}{2 \sin \theta} \int_0^\theta (1 + \cos 2\phi) d\phi \\ &= \frac{\theta}{2 \sin \theta} + \frac{1}{2} \cos \theta, \end{aligned}$$

hence

$$\int_0^1 E(t) dt = G + \frac{1}{2}. \quad (36)$$

A generalisation: the function $\text{Ti}_2(x)$

Going back to the original integral in (1), we can define a function of x by considering the integral on $[0, x]$. The more-or-less standard notation is

$$\text{Ti}_2(x) = \int_0^x \frac{\tan^{-1} t}{t} dt.$$

Clearly, $G = \text{Ti}_2(1)$. Integrating termwise as in (1), we have for $|x| \leq 1$,

$$\text{Ti}_2(x) = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}. \quad (37)$$

This can be compared with “dilogarithm” function $\text{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$. We remark that G appears in the evaluation $\text{Li}_2(i) = -\frac{1}{8}\zeta(2) + iG$.

We can use (37) to calculate values for $|x| \leq 1$, for example $\text{Ti}_2(\frac{1}{2}) \approx 0.48722$.

Since $\tan(\frac{\pi}{2} - \theta) = 1/\tan \theta$ for $0 < \theta < \frac{\pi}{2}$, we have

$$\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$$

for $x > 0$. This translates into a corresponding functional equation for $\text{Ti}_2(x)$:

$$\text{Ti}_2(x) - \text{Ti}_2\left(\frac{1}{x}\right) = \frac{\pi}{2} \log x. \quad (38)$$

It is sufficient to prove this for $x \geq 1$. Write $\text{Ti}_2(x) - \text{Ti}_2(1/x) = F(x)$. Then

$$\begin{aligned} F(x) &= \int_{1/x}^x \frac{\tan^{-1} t}{t} dt = \int_{1/x}^x \frac{1}{u} \tan^{-1} \frac{1}{u} du \\ &= \int_{1/x}^x \frac{1}{u} \left(\frac{\pi}{2} - \tan^{-1} u \right) du \\ &= \pi \log x - F(x). \end{aligned}$$

So for $x > 1$, we have the following series expansion in powers of $1/x$:

$$\text{Ti}_2(x) = \frac{\pi}{2} \log x + \frac{1}{x} - \frac{1}{3^2 x^3} + \frac{1}{5^2 x^5} - \dots,$$

showing very clearly the nature of $\text{Ti}_2(x)$ for large x .

Of course, the procedures that gave equivalent expressions for G can be also applied to $\text{Ti}_2(x)$. For example, the substitution $t = \tan \theta$ leads to

$$\text{Ti}_2(x) = \frac{1}{2} \int_0^{2 \tan^{-1} x} \frac{\theta}{\sin \theta} d\theta.$$

Integration by parts and then again the substitution $t = \tan \theta$ gives

$$\begin{aligned} \text{Ti}_2(x) &= \tan^{-1} x \log x - \int_0^x \frac{\log t}{1+t^2} dt \\ &= \tan^{-1} x \log x - \int_0^{\tan^{-1} x} \log \tan \theta d\theta. \end{aligned}$$

As with the function $\text{Li}_2(x)$, not many other explicit values of $\text{Ti}_2(x)$ are known. One, proved by ingenious methods in [2] (formula (33)) is: $\text{Ti}_2(2 - \sqrt{3}) = \frac{2}{3}G - \frac{1}{12}\pi \log(2 + \sqrt{3})$.

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