A short proof of the identity linking the beta and gamma integrals

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Recall that the *gamma integral* $\Gamma(a)$ is defined, for $a > 0$, by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx \tag{1}$$

and the *beta integral* $B(a,b)$, for a, $b > 0$, by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx. \tag{2}$$

The fundamental identity linking the two is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{3}$$

This is a very well-known identity, and various proofs can be seen in many books. The following particularly short and simple proof appears in [Titch, p. 56].

For any fixed $t > 0$, the substitution $x = ty$ in (1) gives

$$\Gamma(a) = t^a \int_0^\infty y^{a-1} e^{-ty}\, dy \tag{4}$$

Meanwhile, the substitution $x = 1/(1+t)$ in (2) gives

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\, dt. \tag{5}$$

It might seem obvious to express $\Gamma(a)$ and $\Gamma(b)$ by (1) and to work on the resulting double integral expression for $\Gamma(a)\Gamma(b)$. This can be done, using a substitution of the form $y = t - x$ in the integral for $\Gamma(b)$. However, a neater and shorter proof is achieved by using (4) instead of (1) to express one of them.

Proof of (3). Expressing $\Gamma(a)$ by (1) and $\Gamma(b)$ by (4) (with t replaced by x), we have

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \Gamma(b) \int_0^\infty x^{a-1} e^{-x}\, dx \\
&= \int_0^\infty x^{a-1} e^{-x} x^b \left(\int_0^\infty y^{b-1} e^{-xy}\, dy \right) dx \\
&= \int_0^\infty y^{b-1} \left(\int_0^\infty x^{a+b-1} e^{-x(1+y)}\, dx \right) dy. \qquad (6)
\end{aligned}$$

By (4) again,

$$\int_0^\infty x^{a+b-1}e^{-x(1+y)}\,dx = \frac{\Gamma(a+b)}{(1+y)^{a+b}}.$$

So by (5),

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)\int_0^\infty \frac{y^{b-1}}{(1+y)^{a+b}}\,dy = \Gamma(a+b)B(a,b). \qquad \square.$$

The method applies equally to complex a and b with Re a and Re b positive.

Without any direct reference to the beta integral, the proof also establishes Euler's reflection formula $\Gamma(a)\Gamma(1-a) = \pi/(\sin a\pi)$ for $0 < a < 1$. By (6),

$$\Gamma(a)\Gamma(1-a) = \int_0^\infty y^{-a}\int_0^\infty e^{-x(1+y)}\,dx\,dy = \int_0^\infty \frac{1}{y^a(1+y)}\,dy,$$

and it is well known that this integral equals $\pi/\sin a\pi$).

The case $a = \frac{1}{2}$ gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. In this case, the last integral is easily evaluated by the substitution $y = t^2$. Of course, this substitution also shows that $\Gamma(\frac{1}{2})$ equates to the probability integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Reference

[Titch] E.C. Titchmarsh, *The Theory of Functions*, Oxford Univ. Press (1939).

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