CHAOTIC GROUP ACTIONS

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Abstract. We introduce the notion of chaotic group actions and give a preliminary report on their properties. In particular, we show that a group G possesses a faithful chaotic action on some Hausdorff space if and only if G is residually finite. This gives an elementary unified proof of the residual finiteness of certain groups. We also show that the circle does not admit a chaotic action of any group, whilst every compact surface admits a chaotic $\mathbb{Z}$-action.

§1. Introduction.

In recent years an enormous amount of work has been conducted on chaotic dynamical systems. Most of this work has been concerned with the iteration of single maps; in other words, with group (or semi-group) actions of the additive group $\mathbb{Z}$. Now, according to R.Devaney's [D2] definition (see also [BBCDS], [GW] and [Si]), a map is chaotic if it is topologically transitive and if the set of periodic points is dense. The purpose of this present paper is to introduce the analogous notion for actions of arbitrary groups:

Definition. Suppose that a group G acts continuously on a Hausdorff topological space M. Then we say that the action of G on M is *chaotic* if the following two conditions are met:

(a) *topological transitivity:* for every pair of non-empty open subsets U and V of M, there is an element $g \in G$ such that $g(U) \cap V \neq \varnothing$.

(b) *finite orbits dense:* the set of points in M whose orbit under G is finite is a dense subset of M.

Notice that in condition (b) of the above definition, finite orbits for general group actions are a direct generalization of periodic orbits for $\mathbb{Z}$-actions. Phenomena very similar to that of chaotic actions have been studied for decades, though the word "chaos" was not used. For example, as P.Eberlein relates in his description [E] of the work on the geodesic flow in the 1920's: "The object of most of the works in this period was to establish topological dynamical properties of the geodesic flow such as the density of periodic trajectories (=closed geodesics) and the existence of a dense trajectory (topological transitivity)." Now it can be easily verified (cf. [BBCDS], [GW] and [Si]) that a chaotic action of a group G on an infinite metric

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space M is "chaotic" in the popular sense that it has sensitive dependence on initial conditions; that is, there exists $\delta > 0$ such that for every open set U in M there exist $x, y \in U$ and $g \in G$ such that are $g(x)$ and $g(y)$ are at least distance δ apart.

The basic example of a chaotic action is provided by the linear action of $SL(n, \mathbb{Z})$ on the torus $\mathbb{T}^n$, for any $n \geqslant 2$. Condition (b) in the above definition is verified for this action, since the points with rational coordinates have finite orbit. To see that condition (a) is satisfied, one shows that every invariant open subset of $\mathbb{T}^n$ is dense.

In this paper, we provide a collection of observations and questions concerning chaotic group actions. These actions are not merely an artificial generalization of chaotic $\mathbb{Z}$ actions; as we show in Section 3 below, there exist chaotic actions of a group G for which the restriction to every one generator subgroup of G is not chaotic. In our study we do not assume any differentiability or measure theoretic hypotheses and so our results are all quite elementary. Nevertheless, as we hope to convince the reader, the structure is sufficiently rich as to provide a variety of results.

§2. Chaos Equals Residual Finiteness.

Now the two conditions in the above definition of a chaotic action are quite different in nature. The first condition is an irreducibility condition. The second condition is just a disguised form of residual finiteness. Indeed, recall that a group G is said to be residually finite if for every non-identity element g of G, there is a normal subgroup, not containing g, of finite index in G. Then one has:

Theorem 1. *For a group G, the following conditions are equivalent:*

(a) G is residually finite,

(b) there is a faithful action of G with finite orbits dense on some Hausdorff topological space M,

(c) there is a faithful action of G with all orbits finite on some Hausdorff topological space M,

(d) there is a faithful chaotic action of G on some Hausdorff topological space M.

Proof. The proof is particularly simple. Clearly (d) and (c) each imply (b). We show that (b) implies (a) and that (a) implies (c) and (d).

(b $\implies$ a). Suppose that a group G acts faithfully with finite orbits dense on a space M and that g is an element of G, other than the identity element. Since the set F of points of M with finite orbit under G is dense in M, there exists a point $x \in F$ which is not fixed by g. Let $O(x)$ denote the orbit of x under G and let H_x denote the subgroup of G that fixes $O(x)$ point by point. Then clearly H_x is the required normal subgroup of finite index.

(a $\implies$ c). If G is residually finite, then for each non-identity element $g \in G$, there is a normal finite index subgroup H_g that doesn't contain g. Let M_g denote the quotient space G/H_g. Then G acts on M_g by left translation. Now let M be the disjoint union $\coprod_{g \neq id} M_g$, equipped with the discrete topology. Then G acts faithfully on M and every point has finite orbit, by construction.

(a $\implies$ d). If G is finite, then G acts chaotically on itself, with the discrete topology. Suppose that G is infinite and consider the compact product space $\{0,1\}^G$. The natural action of G on $\{0,1\}^G$ is given by

$$g(f)(x) = f(g^{-1}x),$$

for all $g, x \in G$ and $f\colon G \to \{0,1\}$. It is an elementary exercise to show that this action is transitive. Now suppose that G is residually finite. Let $\{x_1, \dots, x_n\}$ be a finite set of distinct elements of G, choose numbers $y_1, \dots, y_n \in \{0,1\}$ and consider the open set U of functions $f\colon G \to \{0,1\}$ for which $f(x_i) = y_i$ for all i. We will show that U contains an element with finite orbit under G. Notice that since G is residually finite, for every pair of distinct elements $a, b \in G$, there exists a finite index normal subgroup H of G such that a and b belong to different cosets of H. It follows that there exists a finite index normal subgroup K of G such that the elements x_i belong to pairwise distinct cosets of K. Now let f be any function which is constant on the cosets of K and which takes the value y_i on the coset containing x_i. So $f \in U$ and clearly f has finite orbit under G. □

The above theorem is quite useful. For instance, it shows that groups which act chaotically and faithfully have no infinite simple subgroups and so, in particular, they cannot themselves be infinite simple groups. In another direction, the group $\mathbb{Q}$ of rational numbers is not residually finite (see for example [We]) and so $\mathbb{Q}$ cannot act chaotically and faithfully. Other useful well known properties of residually finite groups (see for example [LS] and [MKS]) include: Finitely generated residually finite groups are Hopfian; that is, they are not isomorphic to any of their proper quotient groups. Finitely generated residually finite groups have residually finite automorphism groups. Finitely presented residually finite groups have solvable word problem.

In passing, let us remark that the above theorem provides an elementary and unified manner to prove residual finiteness in many cases:

Corollary 1. *The following groups are residually finite:*

(a) the matrix groups $SL(n,\mathbb{Z})$, for all $n > 1$,

(b) countably generated free groups,

(c) quotients of residually finite groups by finite normal subgroups,

(d) subgroups of residually finite groups.

(e) (finite and infinite) direct products of residually finite groups.

(f) wreath products of residually finite groups by finite groups.

Proof of Corollary 1. The statements in Corollary 1 are well known (though we haven't seen (e) in the literature). Part (a) uses the fact that $SL(n,\mathbb{Z})$ acts with finite orbits dense on $\mathbb{T}^n$, as mentioned in the introduction. Parts (c) and (d) follow immediately from the theorem; indeed, it is clear that if a group G acts faithfully with all orbits finite on a Hausdorff space M, then every subgroup of G also acts faithfully with all orbits finite on M. And if H is a finite normal subgroup, then G/H acts faithfully with all orbits finite on the Hausdorff orbit space M/H. To see Part (b), one first recalls that the free group on two generators is a subgroup of $SL(2,\mathbb{Z})$; indeed, by Sanov's theorem (see [LS]), the subgroup of $SL(2,\mathbb{Z})$ generated by the matrices

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is isomorphic to the free group $\mathbb{Z}*\mathbb{Z}$. So by Part (d), $\mathbb{Z}*\mathbb{Z}$ is residually finite. Part (b) then follows from Part (d) again, using the fact that every countably generated free group is a subgroup of $\mathbb{Z}*\mathbb{Z}$. Part (e) is proved as follows: suppose that groups $G_i, i \in I$ act faithfully with all orbits finite on the Hausdorff spaces $M_i, i \in I$ respectively. Now to each space M_i, add an additional isolated element x_i and denote $\tilde{M}_i$ the union $M_i \cup \{x_i\}$. Then we define an action of G_i on $\tilde{M}_i$ by using the action of G_i on M_i and making x_i a fixed point. Clearly G_i acts faithfully with all orbits finite on $\tilde{M}_i$. Now let M denote the subset of the infinite product $\prod_{i\in I} \tilde{M}_i$ composed of all elements $(y_i)_{i\in I}$ for which only finitely many of the y_i are different from x_i. We equip M with the topology induced by the product topology. Then clearly the infinite direct product $\prod_{i\in I} G_i$ acts faithfully with all orbits finite on M.

Finally Part (f) is similar to Part (e); suppose that a group G acts chaotically on a space M and that H is a finite group. Then there is a natural action of the wreath product $G\mathrm{Wr}H$ on the space $M \times H$, where H is given the discrete topology (see [H]). It is easy to see that this action is chaotic. □

The following groups are known to be residually finite: Fuchsian groups [LS], the mapping class groups of compact Riemann surfaces [G], arithmetic groups [Se] and the group of p-adic integers [We]. It would be interesting to find natural chaotic actions of these groups.

§3. Constructions of Chaotic Group Actions.

First recall that there are many examples of chaotic $\mathbb{Z}$-actions; that is, chaotic homeomorphisms. Perhaps the most basic example is that of the Anosov diffeomorphisms of tori and infranilmanifolds (see [Sm], [Mann]); these maps are chaotic since their periodic points are dense [BR] and by Anosov's closing lemma (see for instance [Sh]), they are transitive on their nonwandering set. (The Anosov diffeomorphisms of tori are just the linear hyperbolic maps; that is, linear maps with no eigenvalues on the unit circle.) Similarly, the pseudo-Anosov maps of compact surfaces are also chaotic (see Exposé 9 in [FLP] and the diagrams in [Mañ], pages 111-116).

Let us now give some general results.

Theorem 2. *Consider a Hausdorff space M and the group* $\mathrm{Hom}(M)$ *of homeomorphisms of M. Then one has:*

(a) If there are group inclusions

$$G \leqslant H \leqslant K \leqslant Hom(M)$$

then the action of H on M is chaotic if the actions of G and K on M are chaotic.

(b) If $G \leqslant H \leqslant \mathrm{Hom}(M)$ and G has finite index in H and if the action of G on M is chaotic, then the action of H on M is chaotic.

(c) *If M is locally compact and if* $\mathrm{Hom}(M)$ *is given the compact-open topology, then the action of G on M is chaotic if and only if the action on M of the closure $\bar{G}$ of G in* $\mathrm{Hom}(M)$ *is chaotic.*

Proof. In Part (a), notice that if a point $x \in M$ has finite orbit under K, then x obviously has finite orbit under H. So if the action of K has finite orbits dense, then the action of H has finite orbits dense. On the other hand, if the action of G is topologically transitive, then clearly the action of H is also topologically transitive. So Part (a) holds. Part (b) is similar to Part (a).

In Part (c), again if the action of $\bar{G}$ has finite orbits dense, then the action of G has finite orbits dense. Now suppose that the action of $\bar{G}$ is topologically transitive. Let U and V be two non-empty open subsets of M. Then there exists $g \in \bar{G}$ such that $g(U) \cap V$ is non-empty. Let x be an element of $U \cap g^{-1}(V)$ and let Θ be the open subset of $\bar{G}$ composed of elements that send x into V. Then $g \in \Theta$ and since G is dense in $\bar{G}$, there exists $h \in G \cap \Theta$. So $h(U) \cap V$ is non-empty and hence the action of G is topologically transitive.

Conversely, if M is locally compact, then the natural map $\mathrm{Hom}(M) \times M \to M$ is continuous. So, if a point $x \in M$ has finite orbit under G, then since G is dense in $\bar{G}$, one has that $G(x)$ is dense in $\bar{G}(x)$. Hence $\bar{G}(x)$ is finite. So if the action of G has finite orbits dense, then the action of $\bar{G}$ has finite orbits dense. Finally, if the action of G is topologically transitive, then obviously so too is the action of $\bar{G}$. □

§4. Manifolds That Admit Chaotic Group Actions.

Chaotic homeomorphisms of the 2-dimensional disc can be constructed as follows. Starting with any Anosov diffeomorphism of the torus $\mathbb{T}^2$, one can quotient by the map $\sigma\colon x \mapsto -x$, to obtain a chaotic homeomorphism on the sphere $\mathbb{S}^2$. (This map was used in [Wa], p.140 to show that expansiveness is not preserved under semi-conjugation.) Then, by blowing up the origin to a circle, one obtains a chaotic homeomorphism on the closed disc. Unfortunately this latter homeomorphism is not the identity on the boundary. This can be rectified by making a slight modification of the above construction. Instead of starting with an Anosov diffeomorphism of $\mathbb{T}^2$, one starts with linked twist map [D1] of the torus $\mathbb{T}^2$. A linked twist map is an appropriately chosen composition of Dehn twists. Consider the particular linked twist map $\bar{f}$ defined as follows: by representing $\mathbb{T}^2$ as the square with vertices $(\pm 1/2, \pm 1/2)$, and edges identified in the usual manner, consider the maps $g\colon \mathbb{T}^2 \to \mathbb{T}^2$ and $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$g(x,y) = \begin{cases} (x, y+2x+1/2) & : \text{if } |x| \leqslant 1/4 \\ (x,y) & : \text{otherwise}, \end{cases}$$
$$h(x,y) = \begin{cases} (x+2y+1/2, y) & : \text{if } |y| \leqslant 1/4 \\ (x,y) & : \text{otherwise}. \end{cases}$$

Then set $\bar{f} = g \circ h$. By [D1], the map $\bar{f}$ is chaotic on the set

$$M = \{(x,y) \mid |x| \leqslant 1/4 \text{ or } |y| \leqslant 1/4\}.$$

Figure 1.

Moreover, $\bar{f}$ is the identity on the boundary of M. Now, quotienting by the map $\sigma\colon (x,y) \mapsto (-x,-y)$, one obtains a chaotic homeomorphism f on the disc D^2 and by construction f is also the identity on the boundary.

Using the above map f, one can clearly obtain chaotic homeomorphisms on all closed surfaces (orientable or not); one simple constructs the surface by identifying boundary arcs on the disc in the standard manner and then obtains the required homeomorphism from f, by semi-conjugacy.

Theorem 3. *Every compact surface (with or without boundary) admits a chaotic $\mathbb{Z}$-action (that is, a chaotic homeomorphism).*

Now it is folkloric that the circle admits no invertible chaotic dynamical system. Indeed, we prove that no group acts chaotically on the circle. In fact, one has:

Theorem 4. *No infinite group acts faithfully with finite orbits dense on the circle $\mathbb{S}^1$.*

Proof. It is well known and easy to prove that $\mathbb{S}^1$ admits no chaotic homeomorphism (see for example [Si]). In fact, one has the following elementary Lemma, which we give without proof:

Lemma. *Suppose that ϕ is a orientation preserving homeomorphism of the circle $\mathbb{S}^1$ having dense periodic points. If ϕ has a fixed point, then ϕ is the identity.*

Now, returning to Theorem 4, suppose that a group G acts faithfully with finite orbits dense on $\mathbb{S}^1$. Then the elements of G all have dense periodic points. Let $x \in \mathbb{S}^1$ be a point with finite orbit under the action of G. Now let G_x^+ be the subgroup of G comprised of the orientation preserving elements that fix x. By the above Lemma, G_x^+ consists only of the identity map. Hence, since G_x^+ is a subgroup of finite index in G, we have that G is finite. □

By the classical theory of S.Cairns and J.Whitehead (see [KiSi]), every smooth compact manifold is triangulable and consequently can be constructed from the closed ball by identification of simplices in its boundary. So given the proof of Theorem 3 above, the obvious question is:

Question 1. *Is there a chaotic homeomorphism of the closed 3-ball B^3 which is the identity on the boundary?*

The method used in dimension 2 doesn't seem to generalize to dimension 3. The 3-ball can be obtained by considering the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{T}^3$, by rotations through π about the x, y and z axes. However, the linked twist maps on $\mathbb{T}^3$ are not respected by this action. The ideas in [BFK] may be useful here; this paper shows that every compact manifold of dimension greater than one admits a Bernoulli diffeomorphism. (Bernoulli diffeomorphisms are ergodic and hence transitive, but they do not all have dense periodic points.)

Finally, as promised in the introduction, we give the:

Example. The group $G = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ acts faithfully and chaotically on $\mathbb{T}^2$ in such a way that none of the elements of G act chaotically on $\mathbb{T}^2$ in R.Devaney's sense.

Proof. First, as described above, there exist chaotic homeomorphisms of the closed disc (and hence of the closed square) which are the identity on the boundary. Let f be such a homeomorphism. Now consider $\mathbb{T}^2$ as the unit square with vertices (i, j) with $i, j \in \{0, 1\}$ and with edges identified in the usual manner; that is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Now use the x and y axes to subdivide $\mathbb{T}^2$ into 4 isometric subsquares. Let F be the homeomorphism of $\mathbb{T}^2$ obtained by applying f in each of the 4 subsquares. Let $g\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the translation $g(x, y) = (x + 1/2, y)$. Similarly, define h by $h(x, y) = (x, y + 1/2)$. Then the group $G = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by F, g and h acts chaotically on $\mathbb{T}^2$. But clearly G contains no element which acts chaotically on $\mathbb{T}^2$.

§5. Other Questions.

In this section we present some open questions which we have been unable to resolve. The main question is the following:

Question 2. *Is there a faithful chaotic action of $\mathbb{Z} \times \mathbb{Z}$ on the torus $\mathbb{T}^2$ or the sphere $\mathbb{S}^2$?*

This question is of interest since in order to further the study of chaotic actions, one would naturally look to actions, on low dimensional manifolds, of groups which are simple generalizations of $\mathbb{Z}$. Because of Theorem 4, the obvious place to start is in dimension 2. Now the group $\mathbb{Z} \times \mathbb{Z}$ $(= \mathbb{Z}^2)$ acts chaotically on $\mathbb{T}^4$. But it is not clear whether $\mathbb{Z}^2$ acts chaotically and faithfully on $\mathbb{T}^2$. Notice that $SL(2, \mathbb{Z})$ has no subgroup isomorphic to $\mathbb{Z}^2$. Indeed, $PSL(2, \mathbb{Z})$ is a free product $\mathbb{Z}_2 * \mathbb{Z}_3$ (see [MKS]) and hence by Kurosh's theorem (see [LS]), it cannot have $\mathbb{Z}^2$ as a subgroup.

But $PSL(2,\mathbb{Z})$ is the quotient of $SL(2,\mathbb{Z})$ by the group $\{\pm \mathrm{Id}\} \cong \mathbb{Z}_2$. So $SL(2,\mathbb{Z})$ cannot have $\mathbb{Z}^2$ as a subgroup either.

It follows from the above discussion that if $G = \mathbb{Z}^2$ acts chaotically and faithfully on $\mathbb{T}^2$, then G cannot contain a linear hyperbolic toral automorphism. Indeed, according to [AdPa], if f is a linear hyperbolic toral automorphism and if g is a homeomorphism of $\mathbb{T}^n$ which commutes with f, then g is also a linear toral automorphism. (For more on commuting diffeomorphisms of tori, see [KaSp].)

Another general question is:

Question 3. *What residually finite groups have a faithful chaotic action on some smooth connected compact manifold?*

Clearly finite groups are residually finite but have no faithful chaotic actions on any connected compact manifold. On the other hand, if a group G acts faithfully and chaotically on a compact manifold, then is G necessarily countable?, finitely generated?, discrete as a subgroup of $\mathrm{Hom}(M)$?, closed as a subgroup of $\mathrm{Hom}(M)$? These properties hold for the known examples of chaotic actions constructed from the action of $SL(n,\mathbb{Z})$ on $\mathbb{T}^n$. The properties would seem unlikely to hold in general, but counterexamples have proved to be elusive. Notice that for a smooth compact manifold M, a discrete subgroup $G \leqslant \mathrm{Hom}(M)$ is necessarily countable, since $\mathrm{Hom}(M)$ is second countable. So on smooth compact manifolds one has the following implications:

finitely generated discrete

Hopfian countable closed

Notice that there is a simple partial result: Every topological group acting continuously, faithfully and chaotically on a Hausdorff space is totally pathwise disconnected. To see this, notice that if $G \subseteq \mathrm{Hom}(M)$ acts chaotically, then the only continuous paths in G are the constant paths. Indeed, if γ_t is a path in G and if x has finite orbit under G, then as $\gamma_t(x)$ is a continuous path in M and as $\gamma_t(x)$ belongs to the (finite) orbit of x, so $\gamma_t(x)$ is independent of t. (We remark in passing that it is easy to see that every manifold admits a non-discrete totally pathwise disconnected group of homeomorphisms.)

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