Models for discrete random variables From a mathematical viewpoint any sequence of numbers, $p(r)$, satisfying

$$0 \leq p(r) \leq 1 \quad \text{for all} \quad r \quad \text{and} \quad \sum_{r=0}^{\infty} p(r) = 1,$$

is a valid pmf corresponding to some random variable. However, we are interested only in those random variables which arise from experiments of practical relevance, and thus form plausible models for future statistical modelling. In this Chapter we examine some examples of random variables that result from experiments with well-defined physical mechanisms.

1 Discrete uniform random variables

Consider an experiment where the sample space is $\{0, 1, \ldots, m\}$ and the random variable R corresponds to an outcome being picked at random from the sample space. Each outcome is equi-probable. Examples include:

- random number generators,
- the score on a die,
- the number of heads on the toss a of fair coin.

Exercise 4.1

Write down the pmf, and find the mean value (the expectation) of a discrete uniform rv.

Sol: 4.1

□

Exercise 4.2

Find the expectation of R^2 and consequently the variance of a discrete uniform rv.

Sol: 4.2

□

2 Bernoulli random variables

gqview weblib/Bernoulli.Jacob.html

Jacob Bernoulli (1654-1705) a member of a family of whom as many as 12 have contributed to some branch of mathematics or physics and at least 5 have written on probability. Jacob and his brother John were great rivals and would only communicate in print arguing over the correctness of each other's mathematical proofs.

Consider an experiment where the sample space is $\{0, 1\}$ and the probability of a 1 is θ $(0 \leq \theta \leq 1)$. A random variable R with such a pmf is termed a Bernoulli random variable, examples of which include:

- number of heads on the toss a of biased coin,
- the next patient has cancer,
- the next person smokes and is over 6 ft tall,
- the next baby is a boy.

Here outcomes are sometimes called failure and success for 0 and 1 respectively. For such examples

$$p(1) = \theta, \quad p(0) = 1 - \theta,$$

and $p(r) = 0$ otherwise. The pmf of a Bernoulli random variable R is

$$p(r) = \theta^r (1 - \theta)^{1-r} \quad \text{for} \quad r = 0, 1$$

We say $R \sim \text{Bernoulli}(\theta)$.

End 05.18.4

Note

$$
\begin{aligned}
p(0) &= \theta^0(1-\theta)^{1-0} = 1-\theta, \\
p(1) &= \theta^1(1-\theta)^{1-1} = \theta.
\end{aligned}
$$

Exercise 4.3

Find the mean and variance of a Bernoulli random variable.

Sol: 4.3

□

Notice if $\theta = 0$ or 1 there is no variability, whereas if $\theta = 0.5$ the variability is largest. Is this logical?

3 Binomial random variables

Consider an experiment where the sample space is $\{0, 1, \ldots, n\}$. Here outcomes correspond to the number of successes in n independent Bernoulli trials, each with probability of success being θ. A random variable R with such a pmf is termed a Binomial random variable.

Examples include:

- the number of heads in n tosses of a of biased coin,
- the number of patients with cancer in the next n examined,
- the number of males in a tutorial of size n.

The derivation is a little more complex here so first consider the $n = 3$ case with S and F denoting success and failure respectively. The sample space is

$$\Omega = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

The random variable of interest, R, is the number of successes.

Exercise 4.4

Find $\mathsf{P}(R = r)$ for $r = 0, 1, 2, 3$.

Sol: 4.4

□

Previously, when $\theta = 0.5$, we used equi-probable outcomes to derive the pmf. This is not possible with an arbitrary θ. A general formula which summarises these results is

$$p(r) = \binom{3}{r}\theta^r(1-\theta)^{3-r} \quad \text{for} \quad r = 0, 1, 2, 3.$$

Exercise 4.5

Show that

$$\sum_{r=0}^{3}\binom{3}{r}\theta^r(1-\theta)^{3-r} = 1.$$

Sol: 4.5

□

The more general form for the pmf is: The pmf of a Binomial random variable R is

$$p(r) = \binom{n}{r}\theta^r(1-\theta)^{n-r} \quad \text{for} \quad r = 0, 1, 2, \ldots, n$$

where $0 < \theta < 1$. We say $R \sim \text{Binomial}(n, \theta)$.

This is proved by the following argument:

(i) with r S's and $n-r$ F's $\mathsf{P}(S \dots SF \dots F) = \theta^r(1-\theta)^{n-r}$ by independence.

(ii) P(any sequence of r successes and $n-r$ failures) has this same probability.

(iii) There are $\binom{n}{r}$ possible sequences of r successes and $n-r$ failures.

(iv) Hence $\mathsf{P}(R=r) = \binom{n}{r}\theta^r(1-\theta)^{n-r}$.

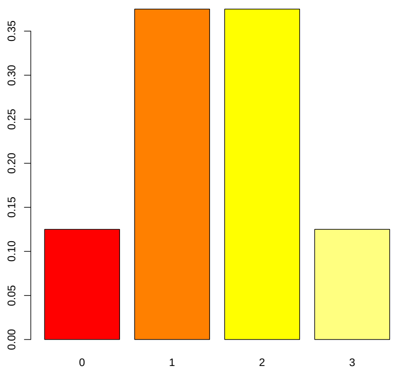
The software package R can evaluate pmfs from standard probability models, including the Binomial.

Exercise 4.6

The rv $R \sim \text{Binomial}(3, 0.5)$. Use R to evaluate and plot the pmf of R. Repeat with $\theta = 0.4$.

```
dbinom(0:3,size=3,prob=0.5)
dbinom(0:3,size=3,prob=0.4)
            # Note how the probabilities change.
p = dbinom(0:3,size=3,prob=0.5)
barplot(p, names.arg=c(0:3))
```

Sol: 4.6



□

Exercise 4.7

Find the probability of rolling a fair die and finding

(i) 2 sixes in 4 rolls,

(ii) 2 sixes in 5 rolls,

(iii) at least 2 sixes in 4 rolls. Note probability of at least 1 six in 4 rolls (Chevalier de Mere) is 0.5177469. In R

```
dbinom(2,size=4,prob=1/6)
```

```
dbinom(2,size=5,prob=1/6)
1-dbinom(0,size=4,prob=1/6)-dbinom(1,size=4,prob=1/6)
1-dbinom(0,size=4,prob=1/6)
```

Sol: 4.7

□

Sol: 4.7

□

Sol: 4.7

□

Exercise 4.8

There are two families each with three children. If each gender has the same probability then find the probability that the families have the same number of girls.

R hint: `sum( dbinom(0:3, size=3, prob=1/2)^2 )`

Sol: 4.8

Let R be the number of girls. Because of independence between children and constant probability for each child model $R \sim \text{Binomial}(3, 0.5)$ For the two families $R_1 \sim \text{Binomial}(3, 0.5)$ and $R_2 \sim \text{Binomial}(3, 0.5)$

□

Expectation and variance

Using the definitions and algebraic manipulation gives For a Binomial random variable $R \sim \text{Binomial}(n, \theta)$

$$\mathsf{E}[R] = n\theta \quad \text{and} \quad \mathsf{E}[R(R-1)] = n(n-1)\theta^2.$$

Hence, by the formula

$$\mathsf{var}(R) = \mathsf{E}[R(R-1)] + \mathsf{E}[R] - (\mathsf{E}[R])^2 = n\theta(1-\theta).$$

The general proof is given in a work sheet solution.

Exercise 4.9

If $R \sim \text{Binomial}(3, \theta)$ show that $\mathsf{E}(R) = 3\theta$.

Sol: 4.9

$$\begin{aligned}\mathsf{E}(R) &= \sum_{r=0}^{\infty} r\, p(r) \\ &= \sum_{r=0}^{3} r\, p(r) \\ &= 0 + \sum_{r=1}^{3} r\, p(r)\end{aligned}$$

Now substitute for the pmf and simplify

$$\begin{aligned}\mathsf{E}(R) &= \sum_{r=1}^{3} r \binom{3}{r} \theta^r (1-\theta)^{3-r} \\ &= \sum_{r=1}^{3} r \frac{3!}{r!\,(3-r)!} \theta^r (1-\theta)^{3-r} \\ &= 3\theta \sum_{r=1}^{3} \frac{2!}{(r-1)!\,(3-r)!} \theta^{r-1} (1-\theta)^{3-r}\end{aligned}$$

Now put $s = r - 1$ and be cunning

$$\begin{aligned} \mathsf{E}(R) &= 3\theta \sum_{s=0}^{2} \frac{2!}{(s)!\,(2-s)!}\theta^s (1-\theta)^{2-s} \\ &= 3\theta \sum_{s=0}^{2} p(s) \\ &= 3\theta. \end{aligned}$$

As $p(s)$ is the pmf of $S \sim \text{Binomial}(2, \theta)$. □

Note that writing $R = \sum_{i=1}^{n} R_i$ where R_i are independent Bernoulli random variables then

$$\mathsf{E}[R] = \sum_{i=1}^{n} \mathsf{E}[R_i] \quad \text{and} \quad \mathsf{var}(R) = \sum_{i=1}^{n} \mathsf{var}(R_i)$$

so in fact it is fairly easy to get these results. These results of linearity of expectation and variance of sums of random variables are proved in MATH 230. End 05.19.1

Exercise 4.10

Is bronchitos infectious and/or hereditary? In a survey of 20 households where both parents had chronic bronchitis, it was found that in 3 of the households the children also developed chronic bronchitis. The probability of developing the disease in any one household is $\theta = 0.05$. To assess how rare this is: find the probability of at least 3 households having children with chronic bronchitis. R hint: `1-sum( dbinom(0:2,size=20,prob=0.05) )`

Sol: 4.10

□

Exercise 4.11

Also find the mean and variance of the number of households having children with chronic bronchitis.

Sol: 4.11

□

4 Geometric random variables

Consider an experiment based on independent Bernoulli trials, each with the probability of a success being θ. Now define the variable of interest, R, to be the number of trials upto BUT NOT including the first success. Here the induced sample space is $\mathcal{S} = \{0, 1, 2, \ldots\}$, and is infinite, corresponding to outcomes in the original sample space

$$\Omega = \{S, FS, FFS, FFFS, \ldots\}.$$

If, for example, the sequence $FFFFS$ occurs then random variable R has the realisation 4.

Such a random variable is called a Geometric random variable, examples of which include:

- the number of heads of a coin toss before the first tail,
- the number of boys born before the first girl,
- the number of black cars passed before a red car,
- the number of league games won before Manchester United first lose.

Exercise 4.12

Use the independence of the Bernoulli random variables to derive the pmf of the geometric random variable.

Sol: 4.12

□

Exercise 4.13

If R is a geometric random variable, with $\theta = 1/2$, find the probability that $R > 1$.

Sol: 4.13

□

Exercise 4.14

Show that $\sum_r p(r) = 1$ for the geometric pmf. This requires the mathematical formulae for sums of geometric type series given in Chapter 1.

Sol: 4.14

□

Exercise 4.15

Find $\mathsf{E}(R)$ for a geometric random variable.

Sol: 4.15

$$
\begin{aligned}
\mathsf{E}(R) &= \sum_{r=0}^{\infty} r\, p(r) \\
&= 0 + \sum_{r=1}^{\infty} r\,(1-\theta)^r \theta \\
&= (1-\theta)\theta[1 + 2(1-\theta)^1 + 3(1-\theta)^2 + \ldots] \\
&= (1-\theta)\theta[(1-(1-\theta))^{-2}] \\
&= \frac{1-\theta}{\theta}.
\end{aligned}
$$

□

Note that as the Bernoulli probability $\theta \downarrow 0$ then the expected number of trials goes to ∞. To summarise For a geometric random variable R

$$
\begin{aligned}
p(r) &= (1-\theta)^r\theta \quad \text{for} \quad r = 0, 1, 2, , \ldots \\
p(0) &= \theta \\
\mathsf{E}[R] &= \frac{1-\theta}{\theta} \quad \text{and} \quad \mathsf{var}(R) = \frac{1-\theta}{\theta^2}.
\end{aligned}
$$

To calculate the variance, first find $\mathsf{E}[R(R-1)]$. End 05.19.2

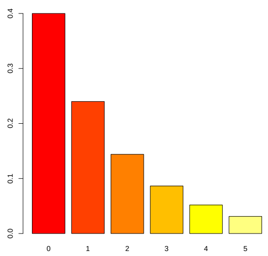
Exercise 4.16

The rv $R \sim \text{Geometric}(0.3)$. Use R to evaluate and plot the pmf of R for $r = 0, 1, 2, \ldots, 5$, with the commands

```
dgeom(0:5,prob=0.3)
dgeom(0:5,prob=0.4)
            # Note how the probabilities change
barplot( dgeom(0:5,prob=0.4),names.arg=c(0:5) )
```

Repeat with $\theta = 0.4$ and plot.

Sol: 4.16



End 04.19.2 □

5 Poisson random variables

gqview weblib/Poisson

The pmf of a Poisson random variable R is

$$p(r) = \frac{\lambda^r \exp(-\lambda)}{r!}, \quad \text{for} \quad r = 0, 1, 2, \ldots$$

where the parameter $\lambda > 0$. We say $R \sim \text{Poisson}(\lambda)$.

Exercise 4.17

Calculate the probabilities of 0, 1 and 2 deaths from typhoid in a year if the number R has a Poisson random variable with $\lambda = 4.6$.

R hint: `dpois(0:2,lambda=4.6)`

Sol: 4.17

□

The Poisson pmf arises physically in two ways:

- The number of events in a fixed time interval of a continuous time process where events occur at random at a given rate over time. (Covered in later courses.)
- The number of successes when the probability of success is very rare.

Examples of Poisson random variables are

- the number of misprints on a page,
- the number of floods of a river in a year,
- the number of deaths due to typhoid over a year, (assuming typhoid cases are independent).

Exercise 4.18

Verify that $p(r) = \frac{\lambda^r \exp(-\lambda)}{r!}$ is a proper probability mass function.

This proof makes use of the definition of the exponential function in terms of its series expansion.

Sol: 4.18

□

Exercise 4.19

If $R \sim \text{Poisson}(\lambda)$ show that $\mathsf{E}(R) = \lambda$.

Sol: 4.19

$$\begin{aligned}
\mathsf{E}(R) &= \sum_{r=0}^{\infty} r\, p(r) \\
&= \sum_{r=0}^{\infty} r\, \frac{\lambda^r \exp(-\lambda)}{r!} \\
&= 0 + \sum_{r=1}^{\infty} r\, \frac{\lambda^r \exp(-\lambda)}{r!} \\
&= \lambda \exp(-\lambda) \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!} \\
&= \lambda \exp(-\lambda) \exp(\lambda) \\
&= \lambda.
\end{aligned}$$

□

Exercise 4.20

Find the variance of the Poisson rv by first computing $\mathsf{E}[R(R-1)]$.

Sol: 4.20

□

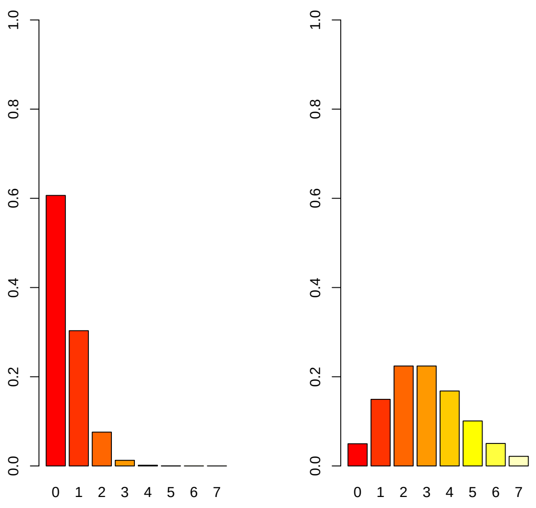
Thus a key property of the Poisson pmf is that the expectation and variance are both equal to λ.

Exercise 4.21

Use R to give a barplot of the Poisson pmf on $0, 1, \ldots, 7$, when $\lambda = 0.5$ and $\lambda = 3$.

```
par(mfrow=c(1,2))
barplot( dpois(0:7,lambda=0.5),names.arg=c(0:7),ylim=c(0,1) )
barplot( dpois(0:7,lambda=3),names.arg=c(0:7),ylim=c(0,1)  )
```

Sol: 4.21



□

Poisson approximation to the Binomial

The Poisson pmf is used for rare event modelling. First we need the following result from calculus:

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x \quad \text{for all} \quad x, \quad [= \exp(x)]$$

Consider a Binomial random variable R from n trials with the probability of success being θ. A rare event has a small probability of occurring, but when there are a large number of trials, the probability that one or more events occur is not negligible.

Define $\lambda = n\theta$.

$$\begin{aligned}
\mathsf{P}(R=r) &= \binom{n}{r}\theta^r(1-\theta)^{n-r} \\
&= \frac{n!}{r!(n-r)!}\left(\frac{\lambda}{n}\right)^r\left(1-\frac{\lambda}{n}\right)^{n-r} \\
&= \frac{\lambda^r}{r!}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to\exp(-\lambda)}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-r}}_{\to 1}\underbrace{\frac{n!}{(n-r)!n^r}}_{\to 1} \\
&\to \frac{\lambda^r}{r!}\exp(-\lambda) \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Rare events: The binomial pmf with large n and small θ can be approximated by a Poisson pmf with parameter $\lambda = n\theta$.

End 05.19.3

For a good approximation $n \geq 100$ and $\theta \leq .01$.

Exercise 4.22

An outbreak of poliomyelitis (Guillain-Barre syndrome =GBS) occurred in Finland in 1984 and as a consequence an intensive vaccination programme occurred. Before the vaccination programme the average number of cases of GBS was 3.67 per month. In the month after the vaccination programme there were 14 cases of GBS reported.

How unusual is this? Compute the p-value = P(observed value or worse), under the assumption of natural Poisson variability. It is $\mathsf{P}(R \geq 14)$ and measures the worry in observing 14 cases.

R hint: `1-sum( dpois(0:13,lambda=3.67) )`

Sol: 4.22

□

People v. Collins

A famous case from 1968.

gqview weblib/Poisson/collins

```
mozilla `pwd`/weblib/Poisson/collins/people.htm
```

The witness used the product rule, which states that the probability of the joint occurrence of a number of mutually independent events is equal to the product of the individual probabilities.

Characteristic probabilities

Characteristic individual probabilities appeared on a table presented in the trial court

A. Partly yellow automobile 1/10,
B. Man with mustache 1/4,
C. Girl with ponytail 1/10,
D. Caucasian girl with blond hair 1/3,
E. Negro man with beard 1/10,
F. Interracial couple in car 1/1000.

Independence implies

$$\mathsf{P}(A \cap B \cap C \cap D \cap E \cap F) = \frac{1}{10} \times \frac{1}{4} \times \frac{1}{10} \times \frac{1}{3} \times \frac{1}{10} \times \frac{1}{1000}$$

$= 8.33 \times 10^{-8}$. 8 chances in 100 million Convict!!!

But

(1) maybe these individual probs are wrong,

(2) maybe these individual events are not independent, though clearly $A \cap B \cap C \cap D \cap E \cap F$ is rare,

(3) but Los Angeles has at least 5 million inhabitants. Of the admittedly few such couples, which one, if any, was guilty of committing this robbery?

Exercise 4.23

Let R denote the number of $A \cap B \cap C \cap D \cap E \cap F$ couples. Argue that $\mathsf{P}(R > 1 | R \geq 1)$ is the probability of interest.

Sol: 4.23

□

Exercise 4.24

Assume that $R \sim \text{Poisson}(\lambda)$ as rare event distribution. Find the probability $\mathsf{P}(R > 1)$ and $\mathsf{P}(R \geq 1)$ in terms of λ.

Sol: 4.24

□

Exercise 4.25

Now find the probability $\mathsf{P}(R > 1 | R \geq 1)$ in terms of λ. Estimate λ from the given information and evaluate the probability that there are other potential suspects.

Sol: 4.25

□

R code:

```
lambda = 10^6*5/(10*4*10*3*10*1000)
( 1-exp(-lambda)-lambda*exp(-lambda))/( 1-exp(-lambda))
```

Are missed lectures Poisson?

Please enter the number of math104/270 lectures you recall missing, on the tally sheet and pass on. To develope a model, introduce notation:
$\surd$ attend lecture, A miss lecture,
R = number of missed lectures in $n = 14$ possibilities. Model R i.e. find a plausible pmf for R.

Make a first assumption: missing lectures are independent , so e.g. $\mathsf{P}(\surd\surd\times) = \mathsf{P}(\surd)\,\mathsf{P}(\surd)\,\mathsf{P}(\times)$. Estimate $\mathsf{P}(\times)$ by 0.25. Theory implies $R \sim \text{Binomial}(14, 0.25)$.

If missing a lecture is rare the Poisson approximation with $\lambda = \quad n\,\mathsf{P}(\times) = 14 \times 0.25$. so that $R \sim \text{Poisson}(3.5)$ should work.

However these models may not work. Suppose that instead of lectures being independent sequences of lecture attendances are independent. Attending lectures is like 'going on a roll', A roll is defined as a sequence of attended lectures ending on an Absence, e.g. $roll = \surd\surd\times$.

Assume (1) The probability of a roll of any length is the same: e.g.

$$\begin{aligned} \mathsf{P}(\surd\surd\times) &= \theta \\ \mathsf{P}(\surd\surd\surd\surd\times) = \mathsf{P}(\times) &= \theta \\ \mathsf{P}(\surd\ldots\surd) = \mathsf{P}(\surd) &= 1-\theta. \end{aligned}$$

(2) Rolls are independent e.g.

$$\mathsf{P}(\surd\surd\times\surd\surd\surd\times\surd\surd) = \mathsf{P}(\surd\surd\times)\,\mathsf{P}(\surd\surd\surd\times)\,\mathsf{P}(\surd\surd)$$

The pmf of the number of rolls is then given by

sequence	value	prob
$\ldots$	0	$1-\theta$
$\ldots\times\ldots$	1	$\theta(1-\theta)$
$\ldots\times\ldots\times\ldots$	2	$\theta^2(1-\theta)$
etc		

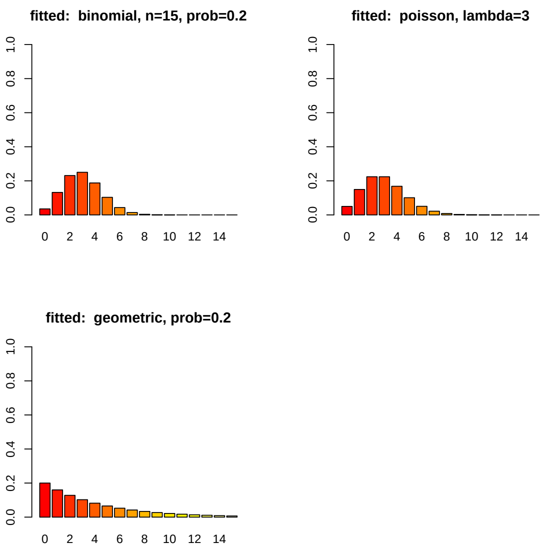
where the $\ldots$ represent any number of $\surd$s This is the $\text{Geometric}(\theta)$ pmf. The number of absences R is then just the number of rolls, $R \sim \text{Geometric}(\theta)$. Again estimate θ by 0.25.

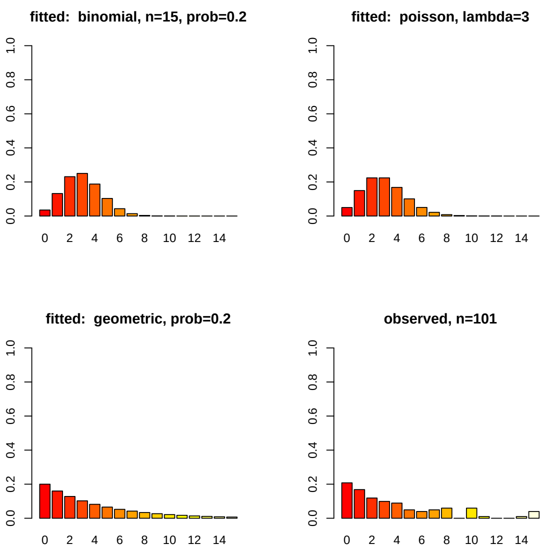
Plot the observed frequencies and fitted values for models: $\text{Binomial}(14, 0.25)$, $\text{Poisson}(3.5)$, $\text{Geometric}(0.25)$.

R code

```
par(mfrow=c(2,3))
r = 0:14
pfit = dbinom(r,size=14,prob=0.25)
  barplot( pfit ,names.arg=r,ylim=c(0,.5))
  title('fitted:  binomial, n=14, prob=0.25')
```

```
pfit = dpois(r,lambda=3.5)
  barplot( pfit ,names.arg=r,ylim=c(0,.5))
  title('fitted:  poisson, lambda=3.5')
pfit = dgeom(r,prob=.25)
  barplot( pfit ,names.arg=r,ylim=c(0,.5))
  title('fitted:  geometric, prob=0.25')
ct05 = c(21,17,12,10,9,5,4,5,6,0,6,1,0,0,1)
ct06 = c(21,10,7,6,14,13,6,3,3,2,1,0,2,2,0 )
pobs = ct05/sum(ct05)
  barplot( pobs  ,names.arg=r,ylim=c(0,.5))
  title('obs 2005, n=97')
pobs = ct06/sum(ct06)
  barplot( pobs  ,names.arg=r,ylim=c(0,.5))
  title('obs 2006, n=90')
```





With data

Observed: gentle decline, go on a roll was the right model in 2005!! Notes: digit preference for [9, 10], worrying tails. End 04.19.3

6 Negative Binomial random variables

Consider an experiment for which the random variable of interest, R, corresponds to the number of fails to occur before the kth success is obtained in a series of independent Bernoulli trials, each with probability of success being θ.

Here the sample space when $k = 2$ is $\{SS, FSS, SFS, FFSS, FSFS, SFFS, \ldots\}$, and the induced sample space is $\{0, 1, 2, \ldots\}$. This is a slightly more general but similar to the geometric random variable case. At might appear the same as Binomial random variable, but the subtle difference is that the last trial must be a success.

For $r = 0, 1, 2, \ldots$

$$\begin{aligned} p(r) &= \mathsf{P}(k-1 \text{ successes in first } k+r-1 \text{ trials, and success on last}) \\ &= \mathsf{P}(k-1 \text{ successes in first } k+r-1 \text{ trials}) \times \mathsf{P}(\text{ success on last}) \\ &= \binom{k+r-1}{k-1}(1-\theta)^{k+r-1}\theta^{k-1} \times \theta \\ &= \binom{k+r-1}{k-1}(1-\theta)^{k+r-1}\theta^{k}. \end{aligned}$$

It is possible to show that

$$\mathsf{E}[R] = \frac{k(1-\theta)}{\theta} \quad \text{and} \quad \mathsf{var}(R) = \frac{k(1-\theta)}{\theta^2}.$$

Note by setting $k = 1$ these results are identical to the geometric pmf.

7 Hypergeometric random variables

Recall the fishing example at the beginning of the course, we now derive the pmf for such a random variable.

- N fish in the lake,
- n are marked and replaced,
- k drawn the second time of which R are marked.

$$\begin{aligned} \mathsf{P}(R = r) &= \frac{\text{ways of choosing r marked and k-r unmarked}}{\text{ways of choosing k}} \\ &= \frac{\binom{n}{r}\binom{N-n}{k-r}}{\binom{N}{k}}. \end{aligned}$$

Properties are more tricky mathematically than the Binomial or Poisson.

Exercise 4.26

Assume that $n = 3$ fish are marked, and that $k = 5$ fish are drawn in the second catch, and there are $N = 7$ fish in total Write down the pmf of R.

Sol: 4.26

$$
\begin{aligned}
p(r) &= \frac{\binom{3}{r}\binom{7-3}{5-r}}{\binom{7}{5}} \\
&\quad \text{for} \quad r = 1, 2, 3 \\
&= 0 \quad \text{otherwise} \\
&= \frac{1}{21}\binom{3}{r}\binom{4}{5-r} \quad \text{for} \quad r = 1, 2, 3
\end{aligned}
$$

□

8 Summary

There are many discrete probability models based on Bernoulli trials (eg coin tossing). The basics are: the sample space $\Omega = \{\omega | \text{seq of H,Ts}\}$, the induced sample space $\mathcal{S}$, usually $\{0, 1, 2, \ldots\}$, independent trials, with constant probability on each trial $\mathsf{P}(H) = \theta$. A random variable R is function from Ω to induced $\mathcal{S}$, with pmf $p(r) = \mathsf{P}(\{R = r\})$. The construction of R determines if the pmf is Bernoulli, Binomial, Geometric etc:

	$\mathcal{S}$	construction	$p(0)$
Bernoulli	$\{0, 1\}$	single throw	$1 - \theta$
Binomial	$\{0, 1, \ldots, n\}$	# Hs in n throws	$(1 - \theta)^n$
Geometric	$\{0, 1, \ldots\}$	# Ts before H	θ
Neg Bino			
Poisson	$\{0, 1, \ldots\}$	Bino limit $n\theta \to \lambda$	$\exp(-\lambda)$
Longest run	$\{0, 1, \ldots\}$	longest run of Hs	0
Max gain	$\{\ldots, -1, 0, 1, \ldots\}$	# Hs − # Ts	?

End 04.19.4