

1 Conditional probability

As we saw in a previous example the probability of an event depends not just on the experiment itself but on other information. Conditional probability forms a framework in which this additional information can be incorporated. If A and B are two events then, as long as $P(B) > 0$, the conditional probability of A given B is written as $P(A|B)$ and calculated from

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Exercise 2.1

If a fair die is thrown and the face shows a number > 2 find the probability that the face shows a prime. Also find the probability of > 2 given it is prime.

Sol: 2.1

□

Exercise 2.2

A bag contains 3 black, 5 white and 2 red marbles. A marble is selected at random. It turns out to be black; find the probability that the next marble selected (without replacing the first) is also black.

Sol: 2.2

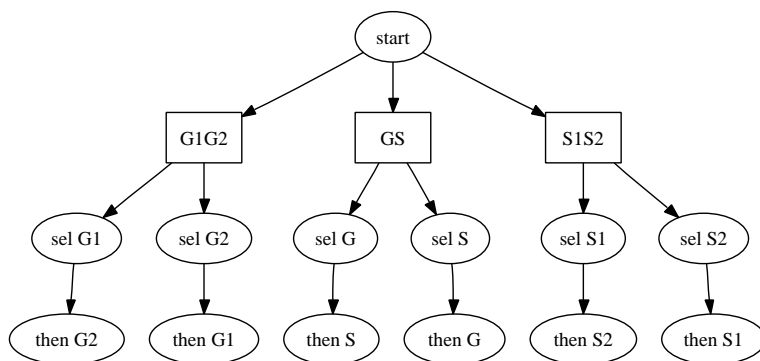
□

Exercise 2.3

Three indistinguishable purses each contain two coins. One purse contains two gold coins, another contains two silver coins and the third contains one gold coin and a silver coin. \boxed{GG} \boxed{GS} \boxed{SS} A purse is selected at random, then at random a coin is selected from it. The selected coin turns out to be gold. Find the probability that the other coin in the purse is also gold.

Sol: 2.3

$$\begin{aligned}
 P(G \text{ left} | G \text{ selected}) &= P(G \text{ selected} \cap G \text{ left}) / P(G \text{ selected}) \\
 &= P(\text{two G purse}) / P(G \text{ selected}) \\
 &= \frac{1/3}{1/2} = 2/3.
 \end{aligned}$$



□

Exercise 2.4

A fair die is thrown giving an odd face; find the probability that it is a 5.

Sol: 2.4

□

Exercise 2.5

Question:

Does $P(A|B)$ satisfy $0 \leq P(A|B) \leq 1$?

Sol: 2.5

□

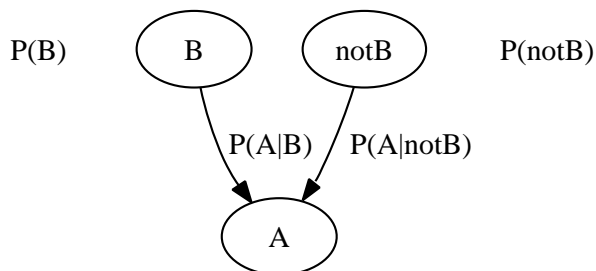
Note: $P(A \cap B) = P(B|A) P(A) = P(A|B) P(B)$.

The partition law can be rephrased in terms of conditional probabilities:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B) P(B) + P(A|B^c) P(B^c), \end{aligned}$$

giving the law of total probability. The law of **total probability**:

$$P(A) = P(A|B) P(B) + P(A|B^c) P(B^c).$$



End 05.17.2

Exercise 2.6

In a population of children 60% are vaccinated against whooping cough. The probabilities of contracting whooping cough are 1/1000 if the child is vaccinated and 1/100 if not. Find the probability that a child selected at random will contract whooping cough.

Sol: 2.6

□

Exercise 2.7

The law of total probability is

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Conjecture a generalisation of the law to find $P(A)$ from a partition $\{B_1, B_2, B_3\}$ of the sample space.

Sol: 2.7

□

Exercise 2.8

Extend the identity $P(A \cap B) = P(A|B)P(B)$ to consider 3 events.

Sol: 2.8

□

Exercise 2.9

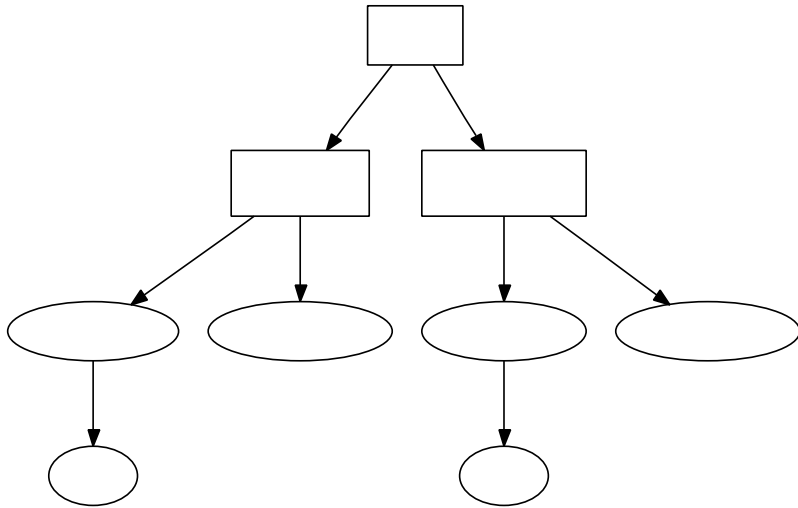
Two urns contain red and blue marbles: Urn A has 5 red, 3 blue, Urn B has 2 red, 1 blue. An experiment takes place in three stages:

Stage 1: Throw a fair die.

Stage 2: If $\{5, 6\}$ occurs select one marble at random from B and transfer it to A ; while if the die shows $\{1, 2, 3, 4\}$ select a marble from A and transfer it to B .

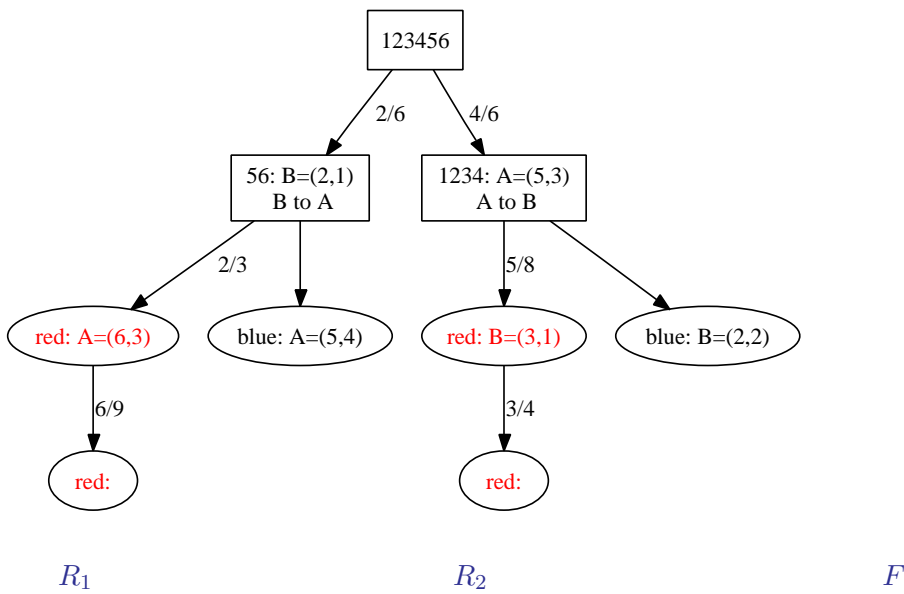
Stage 3: Select a marble from the urn with the extra marble.

Include labels on this diagram that structures the problem.



Find the probability of selecting 2 red marbles. Find the probability of selecting 2 blue marbles.

Sol: 2.9



$$\begin{aligned}
 P(R_2 R_1) &= P(R_2 R_1 | F) P(F) + P(R_2 R_1 | F^c) P(F^c) \\
 &= P(R_2 R_1 | F) \frac{2}{6} + P(R_2 R_1 | F^c) \frac{4}{6} \\
 &= P(R_2 | R_1, F) P(R_1 | F) \frac{2}{6} + P(R_2 | R_1, F^c) P(R_1 | F^c) \frac{4}{6} \\
 &= P(R_2 | R_1, F) \frac{2 \cdot 2}{3 \cdot 6} + P(R_2 | R_1, F^c) \frac{5 \cdot 4}{8 \cdot 6} \\
 &= \frac{6 \cdot 2 \cdot 2}{9 \cdot 3 \cdot 6} + \frac{3 \cdot 5 \cdot 4}{4 \cdot 8 \cdot 6} = 0.461
 \end{aligned}$$

0.461

$$(2/6)(1/3)(4/9) + (4/6)(3/8)(3/4) = 0.174$$

□

Sensitive questions

Using probability to ask sensitive questions.

e.g. 'Have you ever smoked dope?' may not elicit a truthful answer, if asked by the teacher, while it may, if asked by the friend. Get anonymity by randomising the question.

- Roll a die, observe score
- answer Question if score 1,2
- answer Converse Question if score 3,4,5,6.
- $P(Q) = 1/3, \quad P(C) = 2/3.$

Typical questions

Questions for 1,2.

I have smoked cannabis.
 I have stolen from a shop.
 I have cheated in an exam.
 I have cheated on my b/g friend.
 If I saw someone drop £50:
 I would return it.

Converse Questions for 3,4,5,6.

I have never smoked cannabis.
 I have never stolen from a shop.
 I have never cheated in an exam.
 I have never cheated on my b/g friend.
 I would not return it.

Exercise 2.10

Let $p = P(\text{yes}|Q)$ be the conditional probability of answering yes to the Question. Then $P(\text{no}|C) = P(\text{yes}|Q)$ by the structure of the experiment, so $P(\text{no}|C) = p$ as well.

	yes	no
answer Question	p	$1 - p$
answer Converse	$1 - p$	p
Total number	r	$n - r$
	n	

Sol: 2.10

□

Expected proportion saying yes:

$$\begin{aligned} E\left(\frac{R}{n}\right) &= P(\text{yes}) \\ &= \frac{2 - p}{3}. \end{aligned}$$

uses the expectation of the Binomial random variable (see later for this derivation). Hence estimate p by $\hat{p} = 2 - 3\frac{r}{n}$. Tally sheet $\rightarrow r \rightarrow$ estimated p . End 04.17.1

2 Bayes theorem

[gqview weblib/Bayes.html](#)

Thomas Bayes (1702-1761), a clergyman and amateur statistician, was ignored by his contemporaries but has had a profound effect on modern statistical thinking.

Often interest is in the probability of A given B but information is given in terms of the probability of B given A . Bayes theorem however provides the basis for transforming this information. **Bayes theorem** : if A and B are events in the sample space then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

Exercise 2.11

Prove Bayes theorem.

Sol: 2.11

□

Another way to express this theorem is

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

using the law of total probability. To evaluate the right hand side we require the probabilities $P(A|B)$, $P(B)$, $P(A|B^c)$ and $P(B^c)$.

Exercise 2.12

For the whooping cough exercise above, find the probability that a child is vaccinated given the occurrence of whooping cough.

Sol: 2.12

□

Exercise 2.13

The probability of tooth decay, D , is $1/5$ if the child sucks sweets and $1/10$ if not. In the population as a whole the distribution of sweet suckers, S , is $50 : 50$.

- (i) Find the probability that a randomly selected child has tooth decay.
- (ii) If the dentist observes tooth decay find the probability that the child sucks sweets.

Sol: 2.13

□

When more than two possibilities are present, as when $\{B_1, B_2, \dots, B_k\}$ form a partition of the sample space Ω , Bayes formula extends to

$$\begin{aligned} P(B_i|A) &= \frac{P(A|B_i) P(B_i)}{P(A)} \\ &= \frac{P(A|B_i) P(B_i)}{\sum_{i=1}^k P(A|B_i) P(B_i)}. \end{aligned}$$

3 Independent events

Independence is one of the most important concepts in probability. Events A and B are **independent** if and only if

$$P(A|B) = P(A).$$

Independence means that the occurrence of the event B has no effect on the chances that A will occur. An **equivalent** statement to this definition is the multiplication law. The **multiplication** law: A and B are independent events if and only if

$$P(A \cap B) = P(A) P(B).$$

To prove this note $P(A \cap B) = P(A|B) P(B) = P(A) P(B)$ on using the definitions of conditional probability and of independence.

Exercise 2.14

If a coin is thrown twice then the first throw has no effect on the outcome of the second throw. Show the events corresponding to separate outcomes on the 2 throws are independent.

Sol: 2.14

□

Exercise 2.15

The unfair coin. Suppose that the coin is biased and that the probability of a head occurring on any throw is θ , which can be any number between 0 and 1. Use independence to determine the probabilities of the 4 outcomes when throwing the coin twice.

Sol: 2.15

□

Exercise 2.16

Suppose that the probability of mothers being hypertensive (high blood pressure) is 0.1 and fathers is 0.2. Find the probability of a child's parents both being hypertensive, assuming both events are independent.

Sol: 2.16

□

Note: we would expect these two events to be independent if the primary determinants of hypertensivity were genetic, however if the primary determinants were environmental then we might expect the two events not to be independent

Exercise 2.17

Prove from the definition that if A and B are independent then B and A are independent.

Sol: 2.17

□

Exercise 2.18

If $P(A) = 0.2$ and $P(B) = 0.3$ find $P(A \cap B)$ if

- (i) A and B are independent,
- (ii) A and B are exclusive.

Sol: 2.18

□

Exercise 2.19

Consider all mother and their first child blood pressure measurements. Let $A = \{\text{mother's DBP} \geq 95\}$ and $B = \{\text{child's DBP} \geq 80\}$. Suppose we know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B) = 0.05$. Are mother and child's blood pressures independent?

Sol: 2.19

□

4 Failure of networks

We consider a series of examples which involve the calculation of the probability of failure of a network (system) of components.

Each component fails independently with probability θ , or functions with probability $1 - \theta$. A network is a set of such components arranged in series, parallel, or both. The network functions if it is possible to trace a path through functioning components from one end of the network to the other.

Note that the network fails if at least one component fails, equivalently the network functions if all components functions.

Examples of such networks include:

- electrical systems
- telephone networks/exchanges
- management structures.

Exercise 2.20

Find the probability that the network fails when n components are arranged in series.

Sol: 2.20



$$P(\text{netwk fails}) = 1 - P(\text{netwk functions})$$

$$\begin{aligned} P(\text{netwk functions}) &= P(\text{all components function}) \\ &= P(\text{first functions and } \dots \text{ and } n\text{th functions}) \\ &= P(\text{first functions}) \dots P(\text{nth functions}) \\ &= (1 - \theta) \dots (1 - \theta) \\ &= (1 - \theta)^n. \end{aligned}$$

$$P(\text{netwk fails}) = 1 - (1 - \theta)^n$$

□

Independence is a crucial assumption for this calculation.

End 05.17.4

Exercise 2.21

Find the probability that the network fails when n components are arranged in parallel

Sol: 2.21

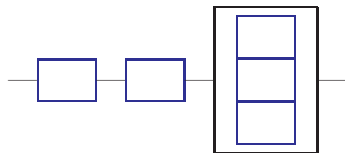


□

Consider components combined in series and parallel.

Exercise 2.22

Find the probability that this network fails.



Sol: 2.22

□

Exercise 2.23

Would you bet that at least two people in your workshop had exactly the same birthday? How large must a workshop be to make the probability of finding two people with the same birthday at least 0.50? Please add your date to the tally sheet. [gqview weplib/exer](#)

Hint R code:

```
options(digits=2)
n = 3 ; prod(366-c(1:n))/365^n #test code
for (n in 20:25){
  p = prod(366-c(1:n))/365^n;
  print(c(n, p))
}
```

Sol: 2.23

2 people. Probability both **different** is $\frac{365}{365} \times \frac{364}{365}$ n students. Probability all different is $\frac{365}{365} \times \frac{364}{365} \dots \frac{365-n+1}{365}$

Empirical results

6 groups: number of pairs / number in group:	group	0/18	0/10	0/13	1/16	0/22	5/44
	Prob all diff	0.653	0.883	0.806	0.716	0.524	0.067

□