Questions for Math 104 Probability

We aim to return the marked scripts within 48 hours so please hand in a fair copy that makes it easy to read and mark. We give out detailed solutions, so marked scripts only provide a mark total, indicate where a solution first goes wrong, and add a little encouragement.

Probabilities, $\mathsf{P}(A)$, are defined on events, A. Very often events are of the form 'variable=value', or 'characteristic=attribute', e.g. $\mathsf{P}(Colour = red)$. So solutions are often of the form $\mathsf{P}(A) =$. Note that e.g. if the right answer is $\mathsf{P}(A) = 0.4$ then 0.4 is insufficient.

Deadlines are indicated by the labels, so that WSwk16 is a question for the workshop in week 16, and CWwk16 is a question to be started in that workshop and handed in by the deadline in week 17.

MATH104 PROBABILITY Workshop Questions SHEET 1.

Q 0.1 *WSwk16 Roads*
There are 6 roads from A to B and 4 from B to C. All roads from A to C go via B.

(a) How many different routes are there from A to C via B?

(b) How many routes are there for the round trip from A to C and back again passing through B on both trips?

(c) How many round trip routes are there that, on the return journey, do not use either of the sections of road used on the outward journey?

Q 0.2 *WSwk16 Two die*
(a) Draw a diagram of the sample space when two distinguishable dice are thrown.

(b) Indicate on the diagram the event that the sum is 7.

(c) Indicate on the diagram the event that the difference is 5.

Q 0.3 *WSwk16 Necklace*
Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and thirty percent wear a necklace.

(a) draw up a 2×2 table to represent this information.

Find the probability that a student is wearing:

(b) a ring or a necklace;

(c) a ring and a necklace;

(d) a ring given they are wearing a necklace.

Q 0.4 *WSwk16 Blackjack*
Two cards are randomly selected from an ordinary playing deck. Find the probability that they form a blackjack, where one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king.

Q 0.5 *WSwk16 Venn*
Draw the Venn diagrams for the following sets:

(a) $A \cup (B \cap C)$,   (b) $(A \cup B) \cap (A \cup C)$,   (c) $(A \cup B)^c$,   (d) $A^c \cap B^c$.

What conclusions do you draw?

Q 0.6 *WSwk16 Journalist*
For discussion in the workshop:
Newspaper columnists recommended that to win the lottery you should

(a) have an equal number of odd and even numbers,

(b) change your numbers each week,

(c) pick numbers which are yet to be drawn, as by the law of averages they are due to occur next,

(d) buy a ticket,

(e) use different numbers for the Wednesday and Saturday draws.

Think about these statements and be prepared to discuss the validity of each.

MATH104 PROBABILITY Assessed Questions SHEET 1.

Q 0.13 *WSwk17 Odd one out*
When three friends go for coffee, they decide who will pay the bill by each flipping a coin at the same time, and then letting the “odd person” pay. If all three flips are the same (so there is no odd person), then they make a second round of flips, and continue to do so until there is an odd person. Find the probability that

(a) exactly 3 rounds of flips are made,

(b) more than 4 rounds are needed?

Q 0.14 *WSwk17 Digits*
The seconds display of a digital watch is randomly distributed on the integers $0, 1, 2, \ldots, 59$. If three watches are independent of each other then find the probability that

(a) all watches display the number 30,

(b) all watches display the same number,

(c) all display different numbers.

Q 0.15 *WSwk17 Rain and snow*
Tomorrow there will be either rain or snow but not both; the probability of rain is $2/5$ and the probability of snow is $3/5$. If it rains then the probability that I will be late for my lecture is $1/5$, while the corresponding probability in the event of snow is $3/5$.

(a) What is the probability that I will be late?

(b) If I am late, what is the probability that there is snow?

Q 0.16 *WSwk17 Visits*
At a psychiatric clinic the social workers are so busy that, on the average, only 60% of potential new patients that telephone are able to talk immediately with a social worker when they call. The other 40% are asked to leave their phone numbers. Of these, about 75% of

the time a social worker is able to return the call on the same day, and the other 25% of the time the caller is contacted on the following day.

Experience at the clinic indicates that the percentage of callers who actually visit the clinic for consultation is 80% if the caller was immediately able to speak to a social worker, whereas it is 60% if the patient's call was returned the same day and 40% if returned on the following day.

First define notation for the events, convert the percentages to probabilities of events, and decide if the probabilities required are conditional or not.

(a) Find the proportion of people that telephone visit the clinic for consultation.

(b) Of the patients that visit the clinic, find the proportion did not have to have their telephone calls returned.

MATH104 PROBABILITY Assessed Questions SHEET 2.

Q 0.21 *WSwk18 Valid pmf*
The discrete random variable takes 3 values with probabilities $(1-\theta)/2$, θ, $(1-\theta)/2$, respectively. Show that for a valid pmf $0 \le \theta \le 1$.

Q 0.22 *WSwk18 P E var*
The rv R has pmf $p(r) = 1/8$ for $r = 1, 2, \ldots, 8$. Find

(a) $\mathsf{P}(R > 2)$, (b) $\mathsf{E}[R]$, (c) $\mathsf{var}(R)$.

Q 0.23 *WSwk18 E var*
Let R denote a random variable that takes the values $1, 2, 3$, with pmf

$$p(r) = \begin{cases} \frac{1}{3} & \text{if} \quad r = 1 \\ \frac{1}{2} & \text{if} \quad r = 2 \\ \frac{1}{6} & \text{if} \quad r = 3. \end{cases}$$

Find $\mathsf{E}[R]$ and $\mathsf{var}(R)$.

Q 0.24 *WSwk18 P even*
The random variable R has pmf $p(r) = (1-\theta)^r\theta$ for $r = 0, 1, \ldots$, find the probability that R is even.

Q 0.25 *WSwk18 Proofs Linearity E*
A random variable R has pmf $p(r)$, for $r = 0, 1, \ldots$. Prove the following results concerning the expectation

(a) $\mathsf{E}[g(R) + h(R)] = \mathsf{E}[g(R)] + \mathsf{E}[h(R)]$,
(b) $\mathsf{E}[cg(R)] = c\,\mathsf{E}[g(R)]$,
(c) $\mathsf{E}[c] = c$,

where g, h are functions and c is constant.

Q 0.26 *WSwk18 Payoff*
There are 2 coins in a bin. When one of them is flipped it lands on heads with probability 0.6, and when the other is flipped it lands on heads with probability 0.3. One of these coins is to be randomly chosen and then flipped. Without knowing which coin is chosen, you can bet an amount up to 10 units and you then either win that amount if the coin comes up heads or lose if it comes up tails.

(i) Calculate the expected gain if you bet x units. It is helpful to sketch the probability tree. You should conclude that it is not in your interest to play the game.

(ii) Suppose, however, that an insider is willing to sell you the information as to which coin was selected for an amount C. Note that if you buy it and then bet x, then you will end up either winning $x - C$ or $-x - C$ (that is, losing $x + C$ in the latter case).

Draw a probability tree to describe this experiment and label the arrows with the conditional probabilities, and label the final outcomes with the winnings. Show the expected gain if you buy this information and if you bet an amount x is $-0.2x - C$.

(iii) If you can vary your bet $0 \leq x \leq 10$ decide for what values of C does it pay to purchase the information,

MATH104 PROBABILITY Assessed Questions SHEET 3.

Q 0.32 *WSwk19 Defective*
A bin of 5 electrical components is known to contain 2 that are defective. If the components are to be tested one at a time, in random order, until both defectives are discovered, find the expected number of tests that are made.

Q 0.33 *WSwk19 Multiple-choice*
A multiple-choice exam with 3 possible answers for each of the 5 questions. Find the probability that a student obtains 4 or more correct answers just by guessing?

Q 0.34 *WSwk19 Abandoned cars*
Suppose that the average number of cars abandoned weekly on a certain highway is 2.2. Assuming that this is a rare event find probability that there will be

(a) no abandoned cars in the next week;

(b) at least 2 abandoned cars in the next week.

Q 0.35 *WSwk19 Geometric E var*
Let R be a geometric random variable with parameter θ. Write down $\mathsf{E}[R]$. Evaluate $\mathsf{E}[R(R-1)]$ and hence $\mathsf{var}(R)$. You may need to inspect the mathematical formulae given in Chapter 1.

Q 0.36 *WSwk19 Beds and pollution*
A small hospital finds that the number of admissions to the emergency ward on a single day ordinarily (unless there is unusually high pollution) follows a Poisson distribution with mean 2.0 admissions per day. Suppose that each admitted person to the emergency ward stays there for exactly 1 day and is then discharged.

(a) The hospital is planning an new emergency-room facility. It wants to have enough beds in the emergency ward so that for at least 95% of normal-pollution days it will have enough beds without turning anyone away. Find the smallest number of beds it should have, to satisfy this criterion.

(b) Previous studies have shown that there is a relationship between admissions and level of pollution on a given day. The hospital also find that on high-pollution days the number of admissions is Poisson distributed with mean 4.0 admissions per day. Again find the smallest number of beds it should have, to satisfy the criterion.

(c) On a random day during the year, find the probability that there are 4 admissions to the emergency ward, assuming that there are 345 normal-pollution days and 20 high-pollution days in every year.

Q 0.37 *WSwk19 Binomial E var*
Let $R \sim \text{Binomial}(n, \theta)$ be a Binomial random variable. Evaluate $\mathsf{E}[R]$ and $\mathsf{var}(R)$.

Q 0.38 *WSwk19 Geometric*
If R is a geometric random variable, find $\mathsf{P}(R \geq n)$. Show that

$$\mathsf{P}(R = n + r \mid R \geq n) = \mathsf{P}(R = r).$$

Give a verbal argument as to why this property should be true.

MATH104 PROBABILITY Assessed Questions SHEET 4.

Q 0.43 *WSwk20 pdf*
A random variable X has probability density function, pdf,

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathsf{E}[X]$ and $\mathsf{var}(X)$.

Q 0.44 *WSwk20 Lifetime*
The lifetime in years of an electronic tube is a random variable X, having a pdf given by

$$f(x) = xe^{-x}, \quad \text{for} \quad x \geq 0.$$

Show that $\mathsf{P}(X > t) = (1 + t)e^{-t}$. Find (a) $\mathsf{P}(X > 5)$, and (b) $\mathsf{P}(X > 10 | X > 5)$.

Q 0.45 *WSwk20 Hala*
Buses for the university via Hala call at a bus station at 15 minute intervals starting at 07.00, whereas buses via A6 call at 15 minute intervals starting at 07.05.

(a) I arrive at the bus station at a time uniformly distributed between 07.00 and 08.00, find the proportion of times I travel via Hala.

(b) Similarly, if I arrive at a time uniformly distributed between 07.10 and 08.10.

Q 0.46 *WSwk20 Chest measurements*
A reasonable model for 19th century Scottish soldiers' chest measurements, X, in inches, is to take $X \sim \text{N}(40, 4)$.

What proportion of the population would have chest measurements between 37 inches and 42 inches inclusive?

Q 0.47 *WSwk20 Repair times*
The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$.

(a) Find the probability that a repair time exceeds 2 hours.

(b) Find the conditional probability that a repair takes at least 10 hours given that its duration exceeds 9 hours.

Q 0.48 *WSwk20 Nicotine*

Blood plasma nicotine levels in smokers can be modelled as $N(315, 131^2)$.

(a) Find the proportion of smokers with nicotine levels lower than 300.

(b) Find the proportion of smokers with nicotine levels between 300 and 500.

(c) If 20 other smokers are to be tested, find the probability that at most one has a nicotine level higher than 500.