1 Previous knowledge

This course assumes that you have covered permutations and combinations, Pascal's triangle and binomial coefficients. The essential results are stated here. You should be familiar with the following terms:

- distinguishable and indistinguishable events
- selection with, or without, replacement
- permutations and factorials
- combinations and binomial coefficients.

Axiom 1: The number of ways of selecting one object from n distinguishable objects is n.

Result 2: The number of ways of selecting one object from n objects and another one from m objects is $n \times m$.

Result 3: The number of permutations of n distinguishable objects is $n!$.

Result 4: The number of permutations of n objects of which r are alike and the remaining $n-r$ are distinguishable is $\dfrac{n!}{r!}$.

Result 5: The number of permutations of n objects composed of r_1 indistinguishable objects of one type and r_2 of another is $\dfrac{n!}{r_1!\,r_2!}$.

Result 6: The number of ways of choosing r objects from n distinguishable objects is $\dfrac{n!}{r!\,(n-r)!}$.

Binomial coefficients

definition

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{for} \quad r = 0, 1, \ldots, n \quad \text{and} \quad n = 1, 2, \ldots$$

We read $\binom{n}{r}$ as n choose r.

Result 7: identity

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Result 8:

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

Result 9:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{n-r}$$

Useful mathematical identities

Arithmetic progression:

$$\sum_{i=1}^{n} i = n(n+1)/2.$$

Sum of squares: result

$$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6.$$

Exponential series:

$$\begin{aligned} \exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad definition \end{aligned}$$

$$\begin{aligned} e &= \exp(1) \quad definition \\ \exp(x) &= e^x \quad result \\ \exp(x+y) &= \exp(x)\exp(y) \quad result \end{aligned}$$

Geometric type sums

Partial geometric sum:

$$\begin{aligned} S_m &= 1 + \theta + \theta^2 + \ldots + \theta^m \\ &= \sum_{i=0}^{m} \theta^i \\ &= \frac{1 - \theta^{m+1}}{1 - \theta} \end{aligned}$$

Geometric sum:

$$\begin{aligned} S_\infty &= 1 + \theta + \theta^2 + \ldots \\ &= \sum_{i=0}^{\infty} \theta^i \\ &= \frac{1}{1 - \theta} \quad \text{for} \quad |\theta| < 1 \end{aligned}$$

Using the result for geometric sums similar results can be derived. Since

$$(1 - \theta)^{-1} = 1 + \theta + \theta^2 + \ldots = \sum_{i=0}^{\infty} \theta^i$$

then by differentiating, term by term, each side with respect to θ gives

Weighted geometric sum:

$$(1 - \theta)^{-2} = 1 + 2\theta + 3\theta^2 + \ldots = \sum_{i=1}^{\infty} i\theta^{i-1}$$

$$2(1 - \theta)^{-3} = 2 + 6\theta + 12\theta^2 + \ldots = \sum_{i=2}^{\infty} i(i-1)\theta^{i-2}$$

Binomial expansion

n an integer result

$$\begin{aligned} (p + q)^n &= (p + q) \ldots (p + q) \\ &= \binom{n}{0} p^n q^0 + \binom{n}{1} p^{n-1} q^1 + \ldots + \binom{n}{i} p^{n-i} q^i + \ldots + \binom{n}{n} p^0 q^n \\ &= \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^i. \end{aligned}$$

End 06.16.1

2 Introduction

Probability, the study of chance, involves mathematics, logic, and considerations of the underlying physical mechanisms that cause variability. In most applications the concepts are easy and the results are consistent with intuition, however in some cases it is more difficult and careful consideration must be given.

The origins of probability theory are associated with two French mathematicians, Pascal (1623-1662) and Fermat (1601-1665). Fermat and Pascal corresponded about a problem which in their solution made use of Pascal's triangle. Pascal invented the first working numerical calculating device in 1645 and the programming language PASCAL is named after him.

The web page `http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians` gives much interesting material.

gqview weblib/Pascal
gqview weblib/Fermat

Example: In 1654 Pascal started a correspondence with Fermat, who wrote most of his work as letters to other mathematicians. The letters between Pascal and Fermat addressed the following problem:

certain gamblers in France wanted to know how to determine the division of stakes in an interrupted game of chance, between two equally skilled players, knowing the scores of the players at the time of interruption, and the number of points needed to win the game.

3 Motivating examples

Example: The probability that

- a fair coin gives a head when tossed is 1/2
- a fair die shows a 6 is 1/6
- an ace is drawn from a deck of cards is

4/52=1/13=0.0769

Exercise 0.1

Notation: Rephrase these answers as statements about probabilities.

Sol: 0.1

□

Example: I play a guessing game with Jack and Jill. Jack selects a fair coin, tosses it, and conceals the up turned face:

- he asks me what is the probability that the coin is showing a head. I answer 1/2, by symmetry .

- Jack shows me the coin. It is a head. He asks me what is the probability that the coin is showing a head. I answer that the probability is 1.
- Jack shows the result of a new toss to Jill while concealing it from me. He asks me what is the probability that the coin is showing a head. How should I now reply this time? $1/2$

This example poses questions concerning the nature of probability. It is not just a physical property of the coin. It is also determined by the experiment and observers. The language of conditional probability goes a long way to solve these conundrums. Consider $\mathsf{P}(H|\text{set up})$ rather than $\mathsf{P}(H)$

Exercise 0.2

Three indistinguishable purses each contain two coins. One purse contains two gold coins, another contains two silver coins and the third contains one gold coin and one silver coin. $\boxed{\text{GG}}$ $\boxed{\text{GS}}$ $\boxed{\text{SS}}$

A purse is selected at random and at random a coin is selected from it. It turns out to be gold. What is the probability that the other coin in that purse is also gold?

Sol: 0.2

□

By giving simple motivating examples the course emphasises how random variables and probabilities occur in practise. Naturally, as this is a first course in probability, these examples are mathematically simple. But as the following example shows they can be of practical relevance.

Example: A lake contains an unknown number N of fish. A sample of n fish is drawn from the lake and each of these fish is marked and replaced. A week later a second sample of k fish is drawn from the lake and is found to include exactly m marked fish. We wish to know how many fish there are in the lake? e.g. $n = 20$, $m = 4$, $k = 32$, suggests $N = 20 \times 8$, is guess. We will not solve this problem in this course, but show in the last section of Chapter 5 how to start building a model that might deliver answers.

Exercise 0.3

The king comes from a family of 2 children. Is the probability that the other child is his sister $1/2$?

Sol: 0.3

□

Example: There are 3 prisoners A, B and C. Two prisoners are to be released. Prisoner A says to the guard: at least one of B and C will be released. Let me know the name of one.

The guard refuses saying, your probability of being released now is $2/3$, but if I tell you B is to be released then your probability of release is $1/2$. Since I don't want to hurt your chances of release I won't tell you.

The language of conditional probability is needed to make progress here. Let A denote 'A is to be released' then $\mathsf{P}(A)$ and $\mathsf{P}(A|B)$ are not the same.

Exercise 0.4

This is a true story (Radio 4, Tue Mar 9 GMT 2004).

An elderly women went to see her doctor, and explained anxiously that she must have breast cancer.

The doctor asked her why, and after saying how bad she felt, she added, 'And on the news last week it said that 1 in 9 women have breast cancer. I live in sheltered housing and none of my eight neighbours has breast cancer'.

Sol: 0.4

□

Exercise 0.5

The Chevalier de Mere bets he can get a 6 in four rolls of a fair die. If he get a 6 in four throws, you give him a franc. If he does not he gives you a franc. Do you want to play?

Sol: 0.5

$\square$

Sol: 0.5

$\square$