An Introduction to Body-Bar Frameworks

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Outline

1. Why Body-Bar?
   - What are they?
   - Available Results
   - Significant Applications

2. Basic Theory
   - Definitions
   - Block Decomposition
   - Tay’s Theorem

3. Extensions
   - Body-Hinge
   - Shared End points
   - Further Extensions
What are they?

A body-bar framework - ‘rigid bodies’ in the given dimension, attached by bars with rotatable vertex attachments.
What are they?

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However this puts them into a harder class to analyze.
General Framework Problems in 3-space

General frameworks in 3-space do not have good characterizations.

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Similar (and larger) problems in higher dimensions $d \geq 4$. 
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Growing number of people doing work within these models.
Significant Applications

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1. Some standard linkages;
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3. Extensions to macromolecules;
4. Control of formations of full dimensional agents.
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1. In 3-space, coordinates are 6-vectors;
2. called Plücker coordinates for the lines;
3. the Cayley algebra 2-extensions for the join of two points on the line $p_i \vee p_j$.
4. the exterior product of two points on the line.
One way to generate the line vector is to place the affine coordinates of the two points as rows of a matrix, and systematically take all the $2 \times 2$ minors of this matrix:

$$\mathbf{R}(G, e) = \begin{pmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \end{pmatrix}.$$ 

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In $d$-space, we generate a $\binom{d+1}{2}$ vector of minors of the corresponding $2 \times d$ matrix. Centers of motion are dual vectors of the same vector size.
Rigidity Matrix

In 3-space, we have such 6-vectors for the constraints and also for the centers of motion $S_i$ for each body. The bar gives a constraint on possible centers of motion for the two bodies as a dot product of the 6-vectors.

$$(\bar{e}_{ij}) \ast (S_i) - (\bar{e}_{ij}) \ast (S_j) = 0$$

or equivalently

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We collect these together as the $|E| \times 6|B|$ rigidity matrix for the body-bar framework $(G, \bar{E})$

$$R(G, \bar{E}) = \{i, j\}$$

$$\begin{pmatrix}
0 & \cdots & 0 & \bar{e}_{ij} & 0 & \cdots & 0 & -\bar{e}_{ij} & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & \\
0 & \cdots & 0 & \bar{e}_{ij} & 0 & \cdots & 0 & -\bar{e}_{ij} & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & \\
\end{pmatrix}.$$
The maximum rank for the rigidity matrix is $6|B| - 6$, since there are always the trivial motions (giving the same center for every body). If we have a maximum independent set of bars, equivalently a minimal rigid set of bars, $\mathbf{R}(G, \bar{E})$ is $(6|B| - 6) \times 6B$. We can partition the columns with first columns of all bodies, then second columns, then ...

$$
\begin{bmatrix}
\vdots & \cdots & \vdots \\
0 & \cdots & (\bar{e}_{ij})_1 & \cdots & 0 \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
&&&&
\end{bmatrix}
$$
If rows are independent, then there is an non-zero $(6|B| - 6) \times (6B - 6)$ determinant (tying down a vertex).

Take a Laplace block decomposition with $|B| - 1$ square submatrices:

$$
\begin{array}{c|c|c}
\vdots & \vdots & \vdots \\
0 & (\tilde{e}_{ij})_1 & 0 \\
\vdots & \vdots & \vdots \\
0 & (\tilde{e}_{ik})_1 & 0 \\
\end{array}
\begin{array}{c|c|c}
\vdots & \vdots & \vdots \\
0 & (\tilde{e}_{ij})_6 & 0 \\
\vdots & \vdots & \vdots \\
0 & (\tilde{e}_{ik})_6 & 0 \\
\end{array}
$$
For each block, we have a copy of the line graph matrix for those edges - up to scalar multiplication by $(\bar{e}_{ij})_m$:

$$
\begin{pmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\
\vdots & & & & & & & \ddots & & & \\
\vdots & & & & & & & & & & \\
\end{pmatrix}
$$

This block has a non-zero determinant (with one column removed) if and only if the induced graph is a spanning tree. We conclude, there is a non-zero term in the determinant only if there are 6 edge-disjoint spanning trees.

A necessary condition for minimal rigidity is that the graph $(B, E)$ partitions into 6 edge-disjoint spanning trees.

Note 6 $|B| - 6 = 6(|B| - 1)$ matches this edge-disjoint tree decomposition.
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Sufficient Condition: Tays Theorem

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This is proven by giving one such realization, based on the 6 edges of a Tetrahedron. Assign all the edges in Tree$_1$ to one edge, and Tree$_2$ to a second edge ... . For a special Tetrahedron, the extensors for the edges of the form $(1, 0, 0, 0, 0, 0), ... (0, 0, 0, 0, 0, 1)$

With this choice, the entire determinant above will only have the one nonzero term in the Laplace block decomposition.
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With this choice, the entire determinant above will only have the one nonzero term in the Laplace block decomposition.

**Theorem [Tay]** A multi-graph $G = (B, E)$ has realizations as infinitesimally rigid body-bar frameworks if and only if the graph contains 6 edge-disjoint spanning trees.

Equivalently, if and only if there is a subgraph $G^* = (B, E^*)$ such that:

(i) $|E^*| = 6|B| - 6$;

(ii) for all subgraphs $G' = (B', E')$ of $G^*$, $|E'| \leq 6|B'| - 6$. 
Tays Theorem

This results and the proof generalizes to all dimensions.

Theorem [Tay] A multi-graph \( G = (B, E) \) has realizations a infinitesimally rigid body-bar frameworks if an only if the graph contains \( \binom{d+1}{2} \) edge-disjoint spanning trees.

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(ii) for all subgraphs \( G' = (B', E') \) of \( G^* \), \(|E'| \leq \binom{d+1}{2} |B'| - \binom{d+1}{2} \).

With this combinatorial characterization of the sparsity condition comes fast algorithms (order \(|B||E|\)), often referred to as the pebble games.

Recall there is no such fast algorithm for generic rigidity of bar and joint frameworks in 3-space (or higher dimensions).
Body-Hinge Structures

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We can consider a body-hinge graph $G = (B, H)$ as a multi-graph $5G$ with 5 bars for each hinge.
Body-Hinge Theorem

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Assume the multi-graph $5G$ contains 6 edge-disjoint spanning trees. Realize this graph with the edges of the 6 edge-disjoint spanning trees on the 6 edges of the tetrahedron.
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Each set of 5 bars from trees at a hinge will be on 5 of the edges of the tetrahedron. This set all meet a single line - which is used for the Hinge. Other edges not in trees are assigned to preserve the hinges.
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Special Body-Hinge graphs come from molecules - with the covalent bonds as hinges. These have an added geometry (the hinges of the body are all concurrent).

The Molecular Theorem (formerly the Molecular Conjecture) shows that even this specialization does not change the characterization: \( 5G \) contains 6 edge-disjoint spanning trees. The same fast algorithms work.
Inductive construction

There are inductive constructions for all generically rigid body-bar frameworks in $d$-space.

Theorem [Frank and Szego]

1. begin with a single body
2. at each further stage, do one of the two following steps:
   (i) add an extra bar;
   (ii) pinch off $k$ edges $0 \leq k < \binom{d+1}{2}$ existing edges.
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![Diagram of inductive construction](image-url)
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Pinching off 0 edges is attaching a new body with $\left(\frac{d+1}{2}\right)$ bars.
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The **Rigid Unit Mode (RUM)** models are an example with corner sharing tetrahedra. This is equivalent to asking that certain sets of 3 bars meet in a point.

What characterization is there for these structures?
**Further Extensions**

1. Generic Global Rigidity. Recall generically redundantly rigid means that removing any one edge leaves a generically rigid body-bar graph.

   **Theorem Body-Bar Global Rigidity** Connelly, Jordan & Whiteley
   A body-bar framework is generically globally rigid in $d$ space if and only if it is redundantly rigid in $d$-space.
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2. **Symmetry:** The analysis of symmetry for bar and joint frameworks has been extended body-bar frameworks by Guest, Schulze and Whiteley.

3. A related analysis should apply to symmetric body hinge frameworks. For the moment, we can analyze them using the associated bar and joint frameworks $G^2$.
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4. **Body-CAD structures**, with other linear or linearized constraints. E.g. Recent paper by Haller, Lee, Sitheram, Streinu & White.
Thanks
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Questions?