The first Littlewood violation

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Abstract. Let $\pi(x)$ denote the number of primes less than or equal to x, and let $\mathrm{li}(x)$ denote the logarithmic integral. There was, in 1914, overwhelming numerical evidence that $\pi(x) < \mathrm{li}(x)$ for all x. In spite of this, Littlewood announced that $\pi(x) - \mathrm{li}(x)$ changes sign infinitely often. So began the search for an upper bound for the first x for which the difference $\pi(x) - \mathrm{li}(x)$ is positive. First, in 1933, came the 1st Skewes number $10^{10^{34}}$. Entering $2,000,000$ Riemann zeros, we prove that there exists x in the interval $[1.3979183 \times 10^{316}, 1.3984775 \times 10^{316}]$ for which $\pi(x) > \mathrm{li}(x)$. This interval is strictly a sub-interval of the interval in Bays & Hudson, and is narrower by a factor of about 10. Joint with K F Chao, www.olimu.com/Riemann/Riemann.htm